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TM 5-856-3, originally EM 1110-345-415,
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2. An asterisk appears before each lin

3. Below is a list of revised or added
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Symbols

LIST O

A	Area of a structural member Tributary area of a structur (sq. ft)
A_B	Area of steel beam cross sec (sq. in.)
A_b	Net area of the rear face of (sq. ft)
A_c	Area of concrete in cross se
A_f	Area of flange of structural Net area of the exterior fro with openings (sq. ft)
A_g	Cross section area of concre
A_{gf}	Gross area of the exterior f (sq. ft)
A_n	Area of each portion of the ture with openings (sq. ft)
A_{of}	Area of openings in the fron (sq. ft)

a'	Distance from edge "a" to
a"	Distance from edge "a" to
B	Peak value of externally a Elastic stiffness coeffici
B_e	Equivalent peak value of t
B_{gX}	Peak value of the effectiv multi-story building
B_g	Peak value of the external
b	Width of flange of steel b Width of compression flang Long side dimension of two-
b_f	Flange width of channel se
b_s	Width of longitudinal web
b'	Width of stem of reinforce Width of flange of steel b Distance from edge "b" to
b"	Distance from edge "b" to
C	Compression force develope Static compression column i

Symbols

c	Distance from neutral axis to Cohesive strength of the soil laboratory testing procedures The clear distance between co
c_o	Velocity of sound in undistur
c_{refl}	Velocity of sound in the regi
D	Diameter of a cylindrical seg Diameter of the smallest long Diameter of spiral core of a
D.L.F	Dynamic Load Factor = x_m/x_s
D.I.F	Dynamic Increase Factor for s
d	Depth of burst of bomb below Height of burst of bomb above Depth from the compression fa of longitudinal tensile reinf Least lateral dimension of re Total depth of steel beam (in Over-all depth of truss (ft) Diameter or width of column c Diameter of circular reinforc
d'	Distance between centroids of doubly reinforced concrete me

F_o	Summation of all external horizo including foundation reactions
F_p	Pressure factor for underground Normal component of total passiv
$F(t)$	Load applied to a structural ele
f	Fiber stress (psi)
f_a	Column critical buckling stress
f_{ad}	Average stress at design level d
f_B	Dynamic yield strength for struc construction only (psi)
f_b	Elastic buckling stress due to b
f_{bd}	Maximum stress due to bending at subjected to axial load and bend
f_{dy}	Dynamic yield strength of steel
$f_{gX}(t)$	Effective load on the g^{th} floor function of time
f_i	Initial static stress (psi)
f_{lv}	Lower static stress (psi)
f	Dynamic yield strength for reinf

Symbols

H_e	Equivalent impulse acting on th
H_{gt}	Total impulse of the external l
H_t	Total impulse of the external l
h	Height (ft) Story height (ft) Column height (ft) Height of a structure above gro Maximum thickness of channel fl
h_c	Clear height of column (ft)
h_g	Height of the g^{th} story of a mu
h_n	Clearing dimension for each por of rectangular structure with o
h'	Clearing height of a closed rec Vertical distance from roof to structure (ft)
h'_b	Weighted average buildup heigh rectangular structure with ope
h'_f	Weighted average clearing heigh a rectangular structure with op
'	Weighted average clearing heigh

KE	Kinetic energy
$(KE)_a$	Kinetic energy of the actual system
$(KE)_e$	Kinetic energy of the equivalent system
K_f	Either load, mass, or load-mass factor for slab fixed on four sides
K_{ij}	($i = 1, 2, 3, \ldots; j = 1, 2, 3 \ldots$) Stiffness influence coefficient (kips/ft)
K_L	Load factor
K_{LM}	Load-mass factor
K_M	Mass factor
K_{Pc}	Normal component of passive pressure coefficient accounting for the cohesive effect of any soil having an internal friction equal to ϕ
K_{Po}	Normal component of passive pressure coefficient for any soil with internal friction angle equal to ϕ and with zero wall friction developed
$K_{P\phi}$	Normal component of passive pressure coefficient for any soil with internal friction angle equal to ϕ and with maximum wall friction developed
K_R	Resistance factor
K_s	Either load, mass, or load-mass factor for slab simply supported on four sides
KT	Kiloton, 1000 tons
K_z	Either load, mass, or load-mass factor for special edge conditions
K_θ	Ratio of the maximum average overpressure to the reflected overpressure existing on an inclined roof
K'	Beam equivalent length coefficient used in the design of steel beam columns
K''	Column length factor used in the design of steel columns and beam columns
k	Spring constant, force required to cause unit deflection of spring (kips/ft) Soil pressure factor (psi)
k_c	Soil constant for underground explosion (psi)
k_E	Effective spring constant (kips/ft)
k_e	Equivalent spring constant (kips/ft)
k_{ep}	Spring constant in the elasto-plastic range
$k_1, k_2, \ldots k_g \ldots$	Spring constants for the first, second, and the g^{th} stories of a multi-story building (kips/ft)
k_g	Simplified notation of $k_{g,g-1}$

k_{gi}	Spring constant of the coupling spring between the g^{th} and the i^{th} floors of a multi-story building (kips/ft)
k'_{gi}	$k'_{gi} = - k_{gi}$
$k_{g1}, k_{g2}, .k_{gi} \ldots, k_{gn}$	Spring constants of the springs connecting m_g with the masses $m_1, m_2, \ldots m_i, \ldots m_n$
L	Span length of beam or truss (ft) Unsupported length of beam (ft) Length of a rectangular structure in the direction of propagation of the blast wave (ft) Length of a cylindrical segment along the cylinder axis (ft) Length of shear wall center to center of column steel Spacing of columns in each direction
L_1	Distance from the outside of the front face to the inside rear face of a rectangular structure with openings (ft)
L'	Distance from the front face of a rectangular structure to a point under consideration on the roof or sides in the direction of propagation of the blast wave (ft) Length between sections of zero and maximum moment being considered (composite beams)
M	Bending moment applied to a section Moment of forces on piles about their centroidal axis Resisting moment of soil on the footing per unit width
M_c	Bending moment at center line of a beam or slab
M_{cn}	Negative resisting moment in column per foot, flat slabs (kip-ft)
M_{cp}	Positive resisting moment in column strip per foot, flat slabs (kip-ft)
M_D	Maximum design moment in a member under axial load, P_D
M_{dy}	Dynamic bending moment
M_{ed}	Elastic dynamic buckling moment
M_m	Maximum bending moment
M_{fa}	Component in a plane perpendicular to edge "a" of the total resisting moment along the fracture lines bounding area "A," two-way slabs (kip-ft)
M_{fb}	Component in a plane perpendicular to edge "b," of the total resisting moment along the fracture lines bounding area B, two-way slabs (kip-ft)
M_{mn}	Negative resisting moment in middle strip per foot, flat slabs (kip-ft)
M_{mp}	Positive resisting moment in middle strip per foot, flat slabs (kip-ft)

M_n	Negative bending moment at support
M_o	Moment of all external forces on structure about axis of rotation "O"
M'_o	Summation of external moments about the point of rotation excluding footing projection moments
M_P	Plastic resisting moment under bending only
M_{Pcn}	Negative plastic resisting moment in column strip per foot, flat slabs (kip-ft)
M_{Pcp}	Positive plastic resisting moment in column strip per foot, flat slabs (kip-ft)
M_{Pfa}	Component of the total plastic bending moment capacity along the fracture line boundary of area "A" which is in a plane perpendicular to edge "a," or the total positive plastic bending moment capacity for a section parallel to edge "a," two-way slabs (kip-ft)
M_{Pfb}	Component of the total plastic bending moment capacity along the fracture line boundary of area "B" which is in a plane perpendicular to edge "b," or the total positive plastic bending moment capacity for a section parallel to edge "b," two-way slabs (kip/ft)
M_{Pm}	Plastic resisting moment at centerline of beam or slab
M_{Pmn}	Negative plastic resisting moment in middle strip per foot, flat slabs (kip-ft)
M_{Pmp}	Positive plastic resisting moment in middle strip per foot, flat slabs (kip-ft)
M_{pos}	Maximum positive bending moment (kip-ft)
M_{Ps}	Plastic resisting moment at support (kip-ft)
M_{Psa}	Total negative plastic bending moment capacity along edge "a," two-way slabs (kip-ft)
M_{Psb}	Total negative plastic bending moment capacity along edge "b," two-way slabs (kip-ft)
M_y	Yield resisting moment (kip-ft)
M^O_P	Plastic resisting moment per unit of width of slab (kip-ft/ft)
M^O_{Pmb}	Plastic positive bending moment capacity per unit width for short span, two-way slabs (kip-ft/ft)
M^O_{Psa}	Plastic negative bending moment capacity per unit width at center of edge "a" for long span, two-way slabs (kip-ft/ft)
M^O_{Psb}	Plastic negative bending moment capacity per unit width at center of edge "b" for short span, two-way slabs (kip-ft/ft)

M_1	Moment at the intersection of the two-linear portions of the P vs curve of a steel beam section (kip-ft)
M_p'	Theoretical plastic resisting moment (kip-ft)
m	Mass per unit length $\left(\text{kip-sec}^2/\text{ft}^2\right)$ Point mass $\left(\text{kip-sec}^2/\text{ft}\right)$ In concrete design: $f_{dy}/0.85\ f'_{dc}$ Total moving mass of structure and earth included between footings Number of fundamental dimensional quantities
m_e	Mass of the equivalent system $\left(\text{kip-sec}^2/\text{ft}^2\right)$
m_r	Mass of structure considered to rotate as well as translate
m_t	Total mass of the element or structural system under consideration $\left(\text{kip-sec}^2/\text{ft}\right)$
m_{te}	Total mass of the equivalent system $\left(\text{kip-sec}^2/\text{ft}\right)$
$m_1, m_2 .. m_g$	Mass of the first, second, and g^{th} floors of a multi-story building
N	Number of non-dimensional parameters Total number of shear connectors required between the points of zero and maximum moment of a composite beam Weighted number of piles in group
N_f	Ratio of the length of fixed-edge perimeter to the total perimeter
N_s	Ratio of the length of simply-supported perimeter to the total perimeter
n	Number of columns in a story Ratio of modulus of elasticity of steel to modulus of elasticity of concrete Number of dimensional variables Number of stories in a multi-story frame
P	Load or force (kips) Total dynamic load on slab (kips) Panel influence factor for shear walls
P_A	Total load on "A" portion of two-way slab (kips)
P_{av}	Average axial load acting on each of several columns of a frame (kips)
P_{back}	Local overpressure on the back face of a buried rectangular structure (psi)
P_B	Total load on "B" portion of two-way slab (kips)
P_c	Compression mode overpressure (psi)
P_{ca}	Uniform compressive pressure applied radially on arch
P_{cr}	Uniform radial pressure that produces buckling of arch

P_{cyl}	Local overpressure normal to the exterior surface of a cylindrical segment (psi)
P_D	Maximum axial load on column with given M_D (kips)
P'_D	Reduced maximum axial load for long columns
P_d	Deflection mode overpressure (psi) Total dynamic axial load
P_{dome}	Local overpressure normal to the exterior face of a spherical dome (psi)
P_e	Equivalent concentrated load for equivalent system (kips)
P_{ed}	Elastic dynamic buckling load (kips)
$P_e(t)$	Equivalent load on an element as a function of time
P_f	Average value of P(t) for the far (or leeward) side of the arch (psi)
P_{front}	Local overpressure on the front face of a buried rectangular structure (psi)
P_g	Peak underground overpressure resulting at a given location from an underground burst (psi)
P_n	Total vertical load (blast plus static) on column Force acting at any time, t_n Average value of P(t) for the near (or windward) side of the arch (psi)
P_p	Dynamic plastic axial load capacity of column
P_{refl}	Reflected shock wave overpressure for angle of incidence of zero degrees (psi)
$P_{r-\alpha}$	Reflected shock wave overpressure for angle of incidence other than zero degrees (psi)
P_s	Overpressure existing in the incident shock wave for any value of $t-t_d$ (psi)
P_{sb}	Overpressure existing in the incident shock wave when $t-t_d = t_b$ (psi)
P_{side}	Local overpressure on the exterior side walls of a rectangular structure
P_{so}	Initial peak incident overpressure (psi)
P_{soi}	Peak overpressure of the shock wave formed in the interior of a structure with openings (psi)
P_{stag}	Stagnation overpressure; the overpressure existing in a region in which the moving air has been brought completely to rest (psi)
P_v	Axial load on the columns of a frame due to the vertical live and dead loads (kips)

(ΣP_v)	The time average of the total vertical load on the column in the time interval $t_{ge} < t < t_{gm}$
$\bar{P}_{back}$	Average overpressure on the exterior back wall of a rectangular structure (psi)
$\bar{P}_{b\text{-}net}$	Average net overpressure acting on back wall of a rectangular structure (psi)
$(\bar{P}_{back})_{max}$	Peak value of the average overpressure on the exterior back wall of a rectangular structure (psi)
$\bar{P}_{end}$	Average overpressure on the closed ends of a cylindrical segment (psi)
$\bar{P}_{f\text{-}net}$	Average net overpressure acting on front wall of a rectangular structure (psi)
$\bar{P}_{front}$	Average overpressure on the exterior of the front wall of a rectangular structure (psi)
$\bar{P}_{i\text{-}front}$	Average overpressure on the interior front wall of a rectangular structure with openings (psi)
$\bar{P}_{i\text{-}refl}$	Reflected shock wave average overpressure in the interior of a rectangular structure with openings (psi)
$\bar{P}_{i\text{-}roof}$	Average overpressure on the interior roof of a rectangular structure with openings (psi)
$\bar{P}_{i\text{-}side}$	Average overpressure on the interior side walls of a rectangular structure with openings (psi)
$\bar{P}_{net}$	Net average horizontal overpressure exerted on a rectangular structure (psi)
$\bar{P}_{r\text{-}net}$	Average net overpressure acting on the roof of a rectangular structure (psi)
$\bar{P}_{roof}$	Average overpressure on the front slope of a gable roof (psi) Average overpressure on the exterior of a rectangular structure (psi)
$\bar{P}'_{roof}$	Peak average overpressure on the front slope of the first gable of a multi-gabled roof (psi)
$\bar{P}_{s\text{-}net}$	Average net overpressure acting inward on sidewall of rectangular structure (psi)
$\bar{P}_{side}$	Average overpressure on the exterior side walls of a rectangular structure (psi)
P_1	Axial load determined by the intersection of the two linear portions of the P vs M curve of a steel beam section (kips)
$P(t)$	Actual load on a structural element as a function of time
p	Ratio of tensile reinforcement in reinforced concrete members to concrete area, A_s/bd
	Uniformly-distributed load intensity

p_o	Critical steel ratio for reinforced concrete member
$\bar{p}$	Equivalent static load for an arch (psi)
p'	Ratio of volume of spiral reinforcement to the volume of the concrete core (out-to-out of spirals) of a spirally reinforced concrete column Ratio of compressive reinforcement in beams to the concrete area, A'_s/bd
Q	Statical moment of the section about the centroidal axis Strength of a shear connector in composite construction Radiant energy on a unit area $\left(cal/cm^2\right)$
q	Unit drag pressure produced by the incident shock wave (psi)
q_d	Dynamic design bearing pressure on soil $\left(kips/ft^2\right)$
q_m	Maximum bearing pressure on soil $\left(kips/ft^2\right)$
q_o	Maximum unit drag pressure produced by the incident shock wave (psi)
R	Radius of a spherical dome Total resistance of structural element or structural system (kips) Dosage of gamma radiation (roentgens) Crater radius (ft)
R_A	Total resistance of "A" portion of two-way slabs (kips)
R_C	Crater radius (ft)
R_c	Horizontal static load resistance of shear wall at first cracking
R_{dc}	Horizontal dynamic load resistance of shear wall at first cracking
R_e	Reynolds number of the high velocity wind in the incident shock wave Resistance of the equivalent system (kips)
R_g	Simplified notation of $R_{g,g-1}$, the resistance function developed between the g^{th} and $(g-1)^{th}$ floors of a multi-story building
$\bar{R}_g$	The average value of R_g in the time interval from 0 to t'_g
R_{gi}	Resistance function developed between the g^{th} and the i^{th} floors of a multi-story building (kips)
R_{gm}	Maximum resistance of the g^{th} story of a multi-story building (kips)
R_{go}	Resistance developed in the spring which connects m_g to the ground
R_{gt}	Total resistance acting on the g^{th} floor mass (kips)
R_m	Maximum resistance developed by a structural system (kips)
R_{mA}	Plastic resistance of the "A" portion of two-way slabs where the edge is fixed (kips)

R_{mB}	Plastic resistance of the "B" portion of two-way slabs where the edge is fixed (kips)
R_{me}	Maximum resistance of the equivalent system (kips)
R_{mf}	Fictitious maximum resistance (kips)
R_n	Resistance of a structural element or system at time, t_n (kips)
R_u	Horizontal ultimate load resistance of a shear wall (kips)
R_y	Yield resistance of the structure (kips)
R_1	Resistance in the elastic range (kips)
R_{1m}	Maximum resistance in the elastic range (kips)
R_{1mA}	Maximum total resistance in the elastic range - "A" portion of two-way slabs (kips)
R_{1mB}	Maximum total resistance in the elastic range - "B" portion of two-way slabs (kips)
$R(t)$	Resistance as a function of time
$R_g(t)$	Resistance of the g^{th} story columns as a function of time
$\left[R_g(t)\right]_{eq}$	Time variation of the resistance of the equivalent single-degree-of-freedom dynamic system for the g^{th} story
R_o	Dosage of nuclear radiation without shielding (roentgens)
$R(x)$	Resistance as a function of displacement
r	Radius of gyration of section (in.) Resistance per unit length of a beam or per unit area of slab Ratio of web reinforcement = A_v/bs Distance from point of explosion (ft) Roentgens Ratio of steel reinforcing placed perpendicular to the steel beam (in composite construction) in excess of that required to carry the slab bending stresses
$\bar{R}_g$	Average value of the resistance of the g^{th} story when the relative displacement between the g^{th} and the $g\text{-}1^{th}$ story is negative
S	Section modulus
SE	Strain energy absorption (kip-ft)
$(SE)_a$	Strain energy absorption of actual structure (kip-ft)
$(SE)_e$	Strain energy absorption of equivalent structure (kip-ft)
S'	Section modulus about the weak axis
s	Coordinate axis at 45° angle to x axis in x-y plane Spacing of stirrups and spacing of ties in reinforced concrete members (in.)

$\bar{s}$	Distance along s axis to centroid of loading (ft)
$\bar{s}_I$	Distance along s axis to centroid of inertia force (ft)
T	Duration of the external load (sec) Resultant tension force (kips)
T_c	Fundamental period of vibration for complete spherical shell of centroid thickness (sec)
T_n	Natural period of oscillation or fundamental period (sec)
T_{nd}	Lowest natural period of circular arch with pinned or fixed ends
T_{ni}	Natural period of the i^{th} mode of oscillation (sec)
T_r	Time or rise of a step-load (sec)
$T_1, T_2 \ldots, T_g$	Time duration of the external loads on the first, second, and g^{th} floors of a multi-story building (sec)
$T_{1X}, T_{2X} \ldots T_{gX}$	Time duration when the effective loads on the first, second, and g^{th} stories are negative (sec)
T'_{gn}	Period of oscillation of a fictitious system representing the g^{th} story (sec)
t	Thickness of concrete slabs (in.) Thickness of flange of beams (in.) Time measured after the arrival of the incident shock wave (sec) Time variable Thickness of deep beam web
t_a	Time of arrival, or time required for the shock wave to travel from the point of explosion to the chosen location (sec)
t_{av}	Average flange thickness of standard steel beam (in.)
t_b	Time required for overpressure on the rear face of a closed rectangular structure to rise from zero to its maximum value (sec)
t_c	Time required to clear the front face of a structure from the reflection effects (sec)
t_d	Time displacement factor; the time required for the shock front to travel from the frontmost element of a structure to the point or surface under consideration (sec)
t_e	Time at which the limiting elastic deflection is reached
t_f	Flange thickness of a WF steel beam (in.)
t_{ge}	Time at which the limiting elastic deflection of the g^{th} story is reached
t_m	Time required for the vortex generated at the front face of a structure to travel a distance L' across the structure (sec) Time required for maximum displacement of element or structure to occur (sec)
t_o	Duration of the positive phase of the incident shock wave (sec)

t_r	Time of rise; time required for the overpressure in the incident shock wave to rise from zero to its maximum value (sec)
t_s	Thickness of stiffener (in.)
t_t	Time required for the shock to pass over the length of the structure (sec)
t_{ur}	Time of rise of an underground overpressure pulse (sec)
t_w	Web thickness of a steel beam or channel (in.)
t_{yp}	Time required to reach yield point of material (sec)
$t_1, t_2 \ldots t_n, t_{n+1}$	Time sequence
$t_{1m}, t_{2m} \ldots t_{gm}$	Time at which the maximum absolute displacements of the first, second and g^{th} floor masses are reached
t'_b	Time required for the overpressure on the exterior rear face of a rectangular structure with openings to rise from zero to its maximum value (sec)
t'_c	Time required to clear the front face of a structure with openings from reflection effects (sec)
t'_2, t'_g	Time at which the relative displacements X_2 and X_g, respectively, are equal to zero
t_n^-	Time immediately before t_n
t_n^+	Time immediately after t_n
t_{lag}	Time lag
Δt	Time interval used in numerical analysis
$\Delta t_n = t_{n+1} - t_n$	A time interval
U_o	Velocity of the incident shock front (fps)
U_{io}	Velocity of the shock front of the shock wave formed in the interior of a rectangular structure with openings (fps)
u	Particle velocity in the incident shock wave (fps) Bond stress per unit of surface area of bar in reinforced concrete design (psi)
V	Dynamic reaction (kips) Total shear (kips) Total vertical force on piles
V_A	Total dynamic reaction along one edge "a," two-way slabs (kips)
V_{av}	Average vertical shear in length, L' (kips)
V_B	Total dynamic reaction along one edge "b," two-way slabs (kips)
V_c	Total column load in flat slab design (kips)
V_m	Maximum shear capacity of web of deep beam section
V_{max}	Maximum vertical shear force (kips)

v	Vortex velocity (fps) Shear stress (psi)
v_{dy}	Dynamic shear yield strength
$v_{g,n} = v_g\left(t_n\right)$	Velocity of the g^{th} floor at time, t_n
v_m	Maximum shearing stress (psi)
$v_n = v\left(t_n\right)$	Velocity at time, t_n
v_n^+	Velocity at t_n^+
v_n^-	Velocity at t_n^-
v_o	Horizontal velocity of axis of rotation "O"
$v\left(t_e\right)$	Velocity of mass at time t_e
v_y	Static shear yield strength of steel (psi) Average unit shear stress in concrete (psi)
$v(t)$	Velocity as a function of time
W	Total energy yield of an atomic bomb expressed KT of TNT required for an equivalent total energy yield Weight (lbs) Total load on element of structure (kips) Work done (ft-lbs) Dynamic peak blast load
W_a	Work done on the actual system (ft-kips)
W_e	Work done on the equivalent system (ft-kips)
W_g	Maximum work done on the g^{th} floor mass (ft-kips)
W_{gm}	Absolute maximum work done on the g^{th} floor mass (ft-lbs)
W_m	Maximum work done on the equivalent system by the equivalent load
W_p	Fictitious maximum work done on the equivalent system
$W(t)$	Work done as a function of time
w	Uniformly distributed load (kips/ft) Length of channel shear connector (in.) Width of front face of rectangular building (ft)
w_1	Weighting factor for vertical and batter piles
X	Relative displacement in a story of a multi-story building (ft)
X_{gm}	Maximum relative displacement in the g^{th} story (ft)
X_{gs}	Relative displacement of the mass g when the average external loads are applied statically
X_1, X_2, X_g	Relative displacement in the first, second, and g^{th} stories of a multi-story building (ft), $X_g = x_g - x_{g-1}$

X_{ge}	Limiting elastic relative displacement for the g^{th} story column (ft)
x	Distance the incident shock wave travels after impinging on the frontmost element of a cylindrical segment or spherical dome to any point being considered Deflection of a structural element or system (ft) Distance of any pile from the centroidal axis
x_c	Collapse deflection
x_e	Limiting elastic deflection (ft)
x_f	Forced solution of a dynamic system for a given external load
x_{ge}	Absolute displacement of the g^{th} floor at time t_{ge}
x_{gm}	Maximum absolute displacement of the g^{th} floor (ft)
$x_{g,n-1}$, $x_{g,n}$, $x_{g,n+1}$	Absolute displacements of the g^{th} floor at time, t_{n-1}, t_n, and t_{n+1}
x_m	Maximum displacement (ft)
x_{n-1}, x_n, x_{n+1}	Displacement at time t_{n-1}, t_n, and t_{n+1}, respectively
x_o	Initial displacement (ft) Horizontal displacement of axis of rotation "0"
x_s	Displacement of an elastic system subjected to the peak load B acting statically (ft)
$x\left(t_e\right)$	Displacement of a mass at time t_e
x_1, x_2, x_g	Absolute displacement of the first, second, and g^{th} floors of a multi-story building (ft)
$\bar{x}$	Distance to centroid parallel to x axis (ft)
$\dot{x}_o$	Horizontal velocity of axis of rotation "0"
$\ddot{x}_o$	Horizontal acceleration of axis of rotation "0"
$\ddot{x}$	Acceleration in the x direction $\left(ft/sec^2\right)$
y	Displacement or deflection of a structural system (ft)
y_a	Deflection of actual element (ft)
y_{ac}	Midspan deflection of actual element (ft)
y_c	Midspan deflection (ft)
y_e	Limiting elastic deflection (ft) Deflection of equivalent system (ft)
y_{ep}	Limiting deflection in the elasto-plastic range (ft)
y_m	Maximum displacement (ft)
y_n	Displacement at time, t_n (ft)

y_r	Vertical distance from axis of rotation "O" to mass centroid of rotating mass, m_r, of structure
y_1	Deflection at midspan of column strip - flat slab (ft)
y_2	Deflection at midspan of middle strip - flat slab (ft)
y_{1m}	Maximum elastic range deflection (ft)
y_{2m}	Maximum elasto-plastic range deflection (ft)
$\bar{y}$	Distance to the centroid parallel to y axis (ft) Vertical distance from axis of rotation "O" to mass centroid of total moving mass, m
$\ddot{y}$	Acceleration in the y direction $\left(ft/sec^2\right)$
$\dot{y}$	Velocity in the y direction (ft/sec)
$\dot{y}_{cl}$	Velocity of displacement of the center of the beam (ft/sec)
y	Increment of displacement (ft)
Z	Plastic modulus of the cross section $\left(in.^3\right)$
Z'	Plastic modulus of the cross section in the weak direction $\left(in.^3\right)$
z	Depth of the resultant compressive force in concrete tee beam
α	Angle of incidence between the normal to the surface and the direction of propagation of the blast wave (degrees) Central angle of the arch Deflection coefficient for two-way slabs Design load ductility reduction factor
α_o	Angular acceleration of structure about axis of rotation "O" $\left(radians/sec^2\right)$
β	Dimensionless ratio = $0.5P_{so}/14.7$ Ductility ratio
δ_b	Maximum bending deflection due to the application of a unit load (ft/kip)
δ_c	Shear wall lateral deflection at first cracking
δ_n	Sub-area clearing factor for rectangular structures with openings
δ_s	Maximum shear deflection due to the application of a unit load (ft/kip)
δ_u	Shear wall lateral deflection at ultimate resistance
δ_y	Deflection at theoretical yield (ft)
θ	Angle of inclination to the horizontal of a gabled roof (degrees) Parameter used to define the location of a point on the surface of a cylindrical segment or a spherical dome (degrees) End rotation of structural element Angular displacement

θ_B	Joint rotation at bottom of column
θ_c	Rotation of beam at midspan (radians)
θ_o	Angular displacement of structure about axis of rotation "O"
θ_T	Joint rotation at top of column
μ	Viscosity of air in the incident shock wave $\left(\text{lb-sec/ft}^2\right)$ Coefficient of friction
ν	Poisson's ratio
ρ	Mass density of air in the incident shock wave $\left(\text{lb-sec}^2/\text{in.}^4\right)$
ρ_s	Mass density of soil $\left(\text{lb-sec}^2/\text{in.}^4, \text{ or lb-sec}^2/\text{ft}^4\right)$
σ_{cr}	Critical buckling stress (psi)
ϕ	Parameter used to define the location of a point on the surface of a spherical dome Phase angle in the transient response (radians) Internal friction angle (degrees)
ϕ_i	Phase angle of the i^{th} mode (radians)
γ	Unit weight of soil (lbs/cu. ft)
ω_o	Angular velocity of structure about axis of rotation "O" (radians/sec)

ENGINEERING AND DESIGN

DESIGN OF STRUCTURES TO RESIST THE EFFECTS OF ATOMIC WEAPONS

PRINCIPLES OF DYNAMIC ANALYSIS AND DESIGN

INTRODUCTION

5-01 PURPOSE AND SCOPE. This manual is one in a series issued for the guidance of engineers engaged in the design of permanent type military structures required to resist the effects of atomic weapons. It is applicable to all Corps of Engineers activities and installations responsible for the design of military construction.

The material is based on the results of full scale atomic tests and analytical studies. The problem of designing structures to resist the effects of atomic weapons is new and the methods of solution are still in the development stage. Continuing studies are in progress and supplemental material will be published as it is developed.

The methods and procedures were developed through the collaboration of many consultants and specialists. Much of the basic analytical work was done by the engineering firm of Ammann and Whitney, New York City, under contract with the Chief of Engineers. The Massachusetts Institute of Technology was responsible, under another contract with the Chief of Engineers, for the compilation of material and for the further study and development of design methods and procedures.

It is requested that any errors and deficiencies noted and any suggestions for improvement be transmitted to HQDA (DAEN-MCE-D) WASH DC 20314. *

5-02 REFERENCES. Manuals - Corps of Engineers - Engineering and Design, containing interrelated subject matter are listed as follows:

DESIGN OF STRUCTURES TO RESIST THE EFFECTS
OF ATOMIC WEAPONS

EM 1110-345-413	Weapons Effects Data
EM 1110-345-414	Strength of Materials and Structural Elements
EM 1110-345-415	Principles of Dynamic Analysis and Design

EM 1110-345-416	Structural Elements Subjected to Dynamic Loads
EM 1110-345-417	Single-Story Frame Buildings
EM 1110-345-418	Multi-Story Frame Buildings
EM 1110-345-419	Shear Wall Structures
EM 1110-345-420	Arches and Domes
EM 1110-345-421	Underground Structures

a. References to Material in Other Manuals of This Series. In the text of this manual references are made to paragraphs, figures, equations and tables in the other manuals of this series in accordance with the number designations as they appear in these manuals. The first part of the designation which precedes either a dash, or a decimal point, identifies a particular manual in the series as shown in the table following.

EM	paragraph	figure	equation	table
1110-345-413	3-	3.	(3.)	3.
1110-345-414	4-	4.	(4.)	4.
1110-345-415	5-	5.	(5.)	5.
1110-345-416	6-	6.	(6.)	6.
1110-345-417	7-	7.	(7.)	7.
1110-345-418	8-	8.	(8.)	8.
1110-345-419	9-	9.	(9.)	9.
1110-345-420	10-	10.	(10.)	10.
1110-345-421	11-	11.	(11.)	11.

b. Bibliography. A bibliography is given at the end of the text. Items in the bibliography are referenced in the text by numbers inclosed in brackets.

c. List of Symbols. Definitions of the symbols used throughout this manual series are given in a list following the table of contents.

5-03 RESCISSIONS. (Draft) EM 1110-345-415 (Part XXIII - The Design of Structures To Resist the Effects of Atomic Weapons, Chapter 5 - Principles of Dynamic Analysis.)

5-04 BASIC PRINCIPLES USED IN DYNAMIC ANALYSIS AND DESIGN. Before discussing the fundamental principles of dynamic analysis, it will be helpful to review briefly the principles used in the analysis of structures under static load. Two different methods are used either separately or concurrently in static analysis; one is based on the principle of equilibrium and the other is based on work done and internal energy considerations.

Under the application of external loads, a given structure will be deformed and internal forces will be set up in its members. In order to satisfy static equilibrium, the vector sum of all the external and internal forces acting on any free body portion of the structure must be equal to zero. For the equilibrium of the structure as a whole, the vector sum of all the external forces and the reaction of the foundation must also be equal to zero. This principle is used in the analysis of statically determinate structures.

The method based on work done and energy considerations is sometimes used when it is necessary to determine the deformation of a structure. In this method, use is made of the fact that the deformation of the structure causes the point of application of the external load to be displaced. The force then does work on the structure. Meanwhile, because of the structural deformations, potential energy is stored in the structure in the form of strain energy. By the principle of conservation of energy, the work done by the external force and the energy stored in the members must be equal. In static analysis, simplified methods such as the method of virtual work and the method of the unit load are derived from the general principle of energy conservation.

In the analysis of statically indeterminate structures, in addition to satisfying the equations of equilibrium, it is necessary to include a calculation of the deformation of the structure in order to arrive at a complete solution of the internal forces in the structure. The methods based on energy considerations, such as the method of least work and the method based on Castigliano's theorems, are generally used.

In the analysis of structures under dynamic loading, basically the same two methods are used, but the load changes rapidly with time and the acceleration, velocity, and hence, the inertial force and kinetic energy are of magnitudes requiring consideration. Thus, in addition to the internal and external forces, the equation of equilibrium includes the inertial force, and the equation of dynamic equilibrium takes the form of Newton's equation of motion, which is given by equation (5.1).

$$\text{(mass) (acceleration)} = \text{external force} - \text{internal force} \qquad (5.1)$$

As for the principle of conservation of energy, the work done must be equal to the sum of the kinetic energy and the strain energy, namely:

$$\text{work done} = \text{kinetic energy} + \text{strain energy} \qquad (5.2)$$

The strain energy in equation (5.2) includes both the reversible elastic strain energy and the irreversible plastic strain energy. Thus the difference between structures under static and dynamic loads is the presence of inertial force [(mass) (acceleration)] in the equation of dynamic equilibrium and of kinetic energy in the equation of energy conservation. Both terms are related to the mass of the structure. Hence, the mass of the structure becomes a very important consideration in dynamic analysis.

In the above discussion, the two methods of analysis are described as if they were independent of each other. This is not the case. The force and energy equations are inter-dependent and are derived from each other. Although these two methods are not independent, the convenience of applying them to a given problem varies greatly depending on the particular problem under consideration. Generally speaking, the force equation is convenient for analysis of structures, and the energy equation is convenient for design.

5-05 DYNAMICALLY EQUIVALENT SYSTEMS. In dynamic analysis, there are only three quantities to be considered. These are, (1) the work done, (2) the strain energy, and (3) the kinetic energy. To evaluate the work done, the displacement at any point on the structure under external distributed or concentrated load must be known. The strain energy is equal to the summation of strain energies in all the structural elements, which may be in bending, compression, shear, or torsion. The kinetic energy involves the energy of translation and rotation of all the masses of the structure. The actual evaluation of these quantities for a given structure under dynamic load would be complicated. However, for practical problems this can be avoided by using appropriate assumptions.

In order to simplify the problem, a given structure is replaced by a dynamically equivalent system. The distributed masses of the given structure are lumped together into a number of concentrated masses. The strain energy is assumed to be stored in a number of weightless springs which do not have to behave elastically. Similarly, the distributed load is replaced by a number of concentrated loads acting on the concentrated masses. Hence, the equivalent system consists merely of a number of concentrated masses joined together by weightless springs and subjected to concentrated loads which vary with time. This concentrated mass-spring-load system is defined as an equivalent dynamic system.

The reduction of a given structure to an equivalent dynamic system involves the principle of dynamic similarity which is merely the requirement that the work done, strain energy, and kinetic energy of the equivalent system must be identical respectively with those of the given structure [2, 7]. The details of this principle and illustrative examples are given in paragraphs 6.13 through 6.24 in EM 1110-345-416.

For the time being it is sufficient to state that any given structure, be it a multi-story building or a structural member such as a beam or column, can be approximated by an equivalent dynamic system for the purpose of dynamic analysis. The discussion in this manual is limited to the analysis and design of equivalent dynamic systems consisting of concentrated masses, springs, and loads.

a. Basic Dynamic System. The simplest dynamic system consists of a concentrated mass supported by a weightless spring and subjected to a concentrated load. This is shown in figure 5.1.

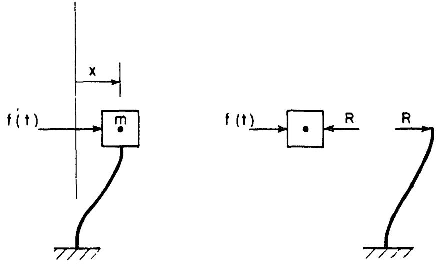


Figure 5.1. Basic dynamic system

In figure 5.1, the following notation is used:

$f(t)$ = external applied load which varies with time

m = mass

R = internal resisting force which depends on the displacement

x = displacement which also varies with time.

The directions in which the variables $f(t)$, R and x are considered to be positive are indicated by arrows in figure 5.1.

Suppose at a given time, t, the displacement, velocity, and acceleration of the mass are x, dx/dt, and d^2x/dt^2 respectively; then, the following expressions may be written:

Internal resisting force = $R = kx$ when the spring is elastic

$$\text{Inertial force} = m\frac{d^2x}{dt^2}$$

$$\text{Work done by the load} = \int_0^x f(t)\,dx = \int_0^t f(t)\frac{dx}{dt}\,dt.$$

$$\text{Kinetic energy} = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$$

$$\text{Strain energy} = \int_0^x R\,dx = \frac{1}{2}kx^2 \text{ when the spring is elastic}$$

Newton's equation of motion is:

$$m\frac{d^2x}{dt^2} = f(t) - R \tag{5.3}$$

and the equation of conservation of energy is

$$\int_0^x f(t)\,dx = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \int_0^x R\,dx \tag{5.4}$$

These are the two basic equations used in dynamic analysis. Before attempting to evaluate the displacement, x, from these equations for a given external load, $f(t)$, a few concepts about the external load and the resistance function need clarification.

(1) External Load and Work Done. Load Duration and Load Impulse. A typical load applied to a dynamic system is shown in figure 5.2. This load acts over an interval of time, T, which is defined as the duration of the load. The impulse of the external load, denoted by H, is equal

to the area under the load-time curve; namely:

$$H = \int_0^T f(t)dt \qquad (5.5)$$

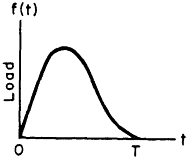


Figure 5.2. A typical dynamic load

Expression for Work Done. The work done by the external load up to any time, t, is given by:

$$W = \int_0^x f(t)dx = \int_0^t f(t)\left(\frac{dx}{dt}\right) dt \qquad (5.6)$$

In order to evaluate the work done, it is necessary first to determine the velocity, dx/dt, which can be obtained from equation (5.3) by integration. This is given in symbolic form as:

$$\frac{dx}{dt} = \frac{1}{m} \int_0^t \left[f(t) - R\right] dt \qquad (5.7)$$

Substituting equation (5.7) into equation (5.6), the expression for work done then is,

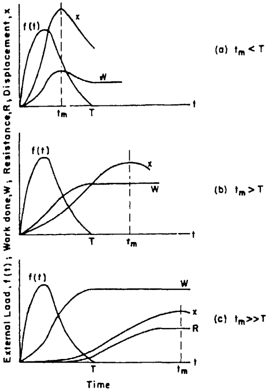
$$W = \int_0^t f(t) \left\{\frac{1}{m} \int_0^t \left[f(t) - R\right] dt\right\} dt \qquad (5.8)$$

Equation (5.8) indicates that the work done by the load varies with time. A typical case is shown in figure 5.3a. At time, t_m, when the displacement of the mass reaches its maximum value, the work done is also a maximum The maximum work done W_m by the external load is obtained when the integral limits in equation (5.8) are from 0 to t_m:

$$W_m = \int_0^{t_m} f(t) \left\{\frac{1}{m} \int_0^t \left[f(t) - R\right] dt\right\} dt \qquad (5.9)$$

Because of the presence of m and R in the integrand, it is evident that the maximum work done depends not only on the external load but also on the mass and resistance function, R, of the dynamic system.

Equation (5.9) is the general expression for the maximum work done. For the cases shown in figures 5.3b and 5.3c, the external load is zero



(a) $t_m < T$

(b) $t_m > T$

(c) $t_m >> T$

Figure 5.3. External load, displacement function and work done curves

over the time interval from T to t_m. The value of W is a constant after T. Hence, for the case where $t_m > T$, the maximum work done is given by:

$$W_m = \int_0^T f(t)\left\{\frac{1}{m}\int_0^t \left[f(t) - R\right] dt\right\} dt \qquad (5.10)$$

Pure Pulse Load. For the case shown in figure 5.3c, the value of t_m is much greater than T and the value of internal resisting force, R in equation (5.10) is negligible compared with the value of the external load over the entire load duration, T. Under this condition, the external load is defined as a pure pulse load. In equation (5.10), if it is assumed that the resistance R is zero for the full load duration, the maximum work done by a pure pulse load is given by:

$$W_m = \int_0^T f(t)\left\{\frac{1}{m}\int_0^t f(t)\, dt\right\} dt = \frac{H^2}{2m} \qquad (5.11)$$

where

$$H = \int_0^T f(t)dt = \text{the impulse of the external load.}$$

A pure pulse load corresponds to an impact load. During impact, the internal resisting force and the strain energy are both assumed to be zero. After impact, the mass, m, acquires an initial velocity equal to H/m. The kinetic energy of the mass after impact is given by equation (5.11) which is also the maximum work done by the load. This equation indicates that for a pure pulse load, the maximum work done depends on the load impulse, but is independent of the detailed shape of the external load.

Fictitious Maximum Work Done. In many actual problems, the internal resisting force, R, in the time interval from 0 to T is not small enough to be neglected. From equations (5.3) and (5.10), it can be seen that if the internal force in the time interval from 0 to T is neglected, the actual acceleration, velocity, and displacement will always be larger than the corresponding values when the resistance is considered. Likewise, the internal force in equation (5.9) always acts to reduce the work done by the external load. In other words, the actual maximum work done on a dynamic system will always be smaller than the maximum work done on the same system where the load is considered to be a pure pulse load. Hence, the work done as computed from equation (5.11) is defined as the fictitious maximum work done, namely,

$$W_p = \frac{H^2}{2m} \tag{5.12}$$

The maximum work done, W_m, as computed from equation (5.9) is always smaller than W_p. The ratio W_m/W_p depends on m, R, and on the detailed shape of the external load-time curve. The work done ratio is an important non-dimensional parameter to be considered in the design of dynamic systems.

(2) Spring Constant, Resistance and Strain Energy. From paragraph 5-05a, it is seen that the function of the internal force is to resist the movement of the mass, hence the internal force is defined as the resistance of the structure. The variation of the resistance with displacement of the mass is expressed by a resistance function.

In figure 5.1, the strain energy is assumed to be stored in a spring. For a linearly elastic spring, the resistance function is equal to kx, and, the strain energy is $\frac{1}{2} kx^2$. For the general case, the strain energy is equal to $\int_0^x R dx$. Then the spring constant, resistance function, and strain energy are related to one another by the following two equations:

$$\frac{d(SE)}{dx} = R \tag{5.13}$$

$$\frac{dR}{dx} = k \qquad (5.14)$$

where

SE = internal strain energies which include both the reversible elastic strain energy and the irreversible plastic strain energy

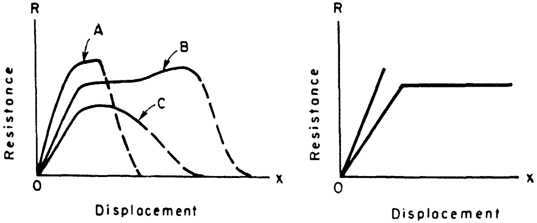
R = resistance

k = spring constant

Equations (5.13) and (5.14) are obtained from the basic dynamic system shown in figure 5.1. For complicated cases when more than one displacement is necessary to specify the configuration of the dynamic system, these two equations still hold true except partial derivatives are used.

The relationship between resistance and strain energy as expressed by equation (5.13) indicates that two different methods may be used to evaluate the resistance function. One method is based on the direct evaluation of internal resisting force. The other method is based on the evaluation of strain energy, and the resistance is then derived from equation (5.13).

The resistance when plotted against the displacement, x , usually takes one of the forms shown in figure 5.4a. For structures made of brittle materials, the resistance function is generally as indicated by curve A. For structures made of ductile materials with marked yielding, the



(a). Typical Resistance Functions (b) Idealized Resistance Functions

Figure 5.4. Resistance functions

resistance function is given by curve B. For curve C, the slope of the resistance function is negative for large values of x . This is generally not a property of the material used, but is due to: (1) the combination of axial and lateral load on a given member of a given structure, and/or (2) the effect of elastic or inelastic instability. In evaluating the response of structures under dynamic load, if better accuracy in the result is desired, the resistance function as shown in figure 5.4a should be used. But the method of preliminary design is usually based on a number of design charts which are evaluated on the basis of idealized resistance functions such as shown in figure 5.4b where one or two straight lines are used to approximate the actual resistance function. The type of idealized resistance to be used is discussed in paragraph 5-07.

b. Degree of Freedom of Dynamic Systems. The basic dynamic system considered in the preceding paragraph consists of one mass, one spring and a concentrated load. Hence, a single displacement variable, x , is sufficient to describe its motion. This basic system is called a single-degree-of-freedom dynamic system, or a single-degree dynamic system.

For the case of a multi-story building, more than one displacement variable is needed to describe its motion. The equivalent system consists of several concentrated masses connected by springs.

The degree of freedom of a dynamic system is defined as the number of independent displacement variables needed to specify completely the configuration of the system. Based on this definition, the dynamic systems shown in figures 5.5a and b have single, and two degrees of freedom, respectively.

For ordinary building structures under lateral loads, the mass of the structure can be assumed to be concentrated at the floor levels and at the roof. The framework between floors can be assumed to be

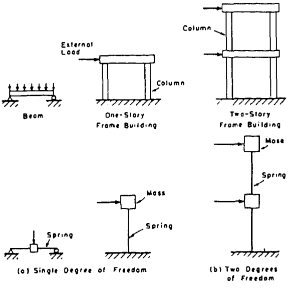


Figure 5.5. Single- and two-degree-of-freedom dynamic systems

weightless splines or flat springs. The degree of freedom of a building is therefore equal to the number of stories of the building. In certain special cases, the displacements of any two floors can be assumed to always bear a constant ratio. The degree of freedom is then reduced. A multistory building may even be approximated by a single-degree dynamic system. However, the error involved in this simplification may be very large. It is not recommended unless it is only for a first approximation in analysis or design.

Beams, columns, slabs, etc., which have a continuous distribution of mass as shown in figure 5.5a are actually infinite-degree dynamic systems. However, if it is assumed that the deflected shapes of the member at any two instants of time are geometrically similar to each other, only one displacement is necessary (for example, the deflection at midspan). Thus, an infinite-degree-of-freedom system is reduced to a single-degree dynamic system. The method of evaluating the parameters of the equivalent dynamic system for these members is discussed in paragraphs 6-13 through 6-24, EM 1110-345-416.

c. Difficulties in Application of Dynamic Systems. A basic difficulty in dynamic analysis results from the fact that the method of superposition is not directly applicable. If there are several loads applied simultaneously on a dynamic system with elastic springs, the maximum values of the displacements produced by individual loads do not generally occur at the same time. The maximum value of the displacement produced by all the loads is not equal to the summation of the maximum values produced by individual loads. Hence, in order to apply the method of superposition, the so-called phase differences in the dynamic responses to individual loads must be considered. Moreover, in some cases, the equivalent dynamic system of a structure may be non-linear, for example when a certain member is stressed beyond the yield point or when there is a vertical load as well as a lateral load acting on the structure. For non-linear dynamic systems, the method of superposition is not applicable at all. Hence, in general, all the loads acting on a dynamic system have to be considered simultaneously.

In dynamic analysis, a given structure is approximated by its equivalent dynamic system. Any structure is made up of a number of elements such as beams, columns, girders, and structural connections. Each member is a distributed mass and possesses an infinite degree of freedom. Rigorously speaking, any structure is a multi-infinite degree dynamic system. However, by making certain assumptions and by applying the principles of dynamic similarity, a complicated structure is simplified to an equivalent dynamic system with a finite degree of freedom. Essentially, a distributed mass-load-spring system is replaced by a lumped mass-load-spring system. The accuracy of the results obtained from the equivalent system as compared with those obtained from the actual structure depends entirely on the accuracy of the assumption used in the derivation of the equivalent system. In some cases, it may involve large error. It should be understood that the replacement of an actual structure by an equivalent dynamic system is one of the basic approximations used in dynamic analysis.

Even for those cases when the equivalent system is acceptable as a good approximation for the given structure, there are difficulties in the evaluation of its parameters. One typical example is the evaluation of the resistance of a one-story frame building. The internal force which resists the lateral movement of the roof is equal to the total shear force at the top of the columns and this shear force is developed because of the bending and shear deformations of the columns. The bending deformation, hence the resistance, depends on the degree of fixity of the ends as well as on the size of the columns. The determination of the end fixity involves the size of girders, foundations, and the type of end connections. In practical problems, for the sake of simplicity, the two extreme conditions of fixity are generally used, that is, the end is assumed either hinged or fixed. For the determination of the resistance function of a member with stresses exceeding the elastic limit, it is necessary to consider the variation of the elastic limit with the rate of strain which is discussed in paragraph 4-03a, EM 1110-345-413. There are still other effects to be considered such as the effect of dead weight and vertical load on the roof. Because of these factors, the actual resistance function is a complicated function of the displacement. Although in the analysis of structures, these factors may be partially taken into account in order to obtain a resistance function which is a closer approximation to the actual resistance function, idealized resistance functions are always used in design.

The same difficulty is encountered in the determination of the load as in the determination of the resistance function. The load is determined either directly or indirectly from the blast pressure which is generally a complicated function of time. Here again, the actual variation of load can be taken into account in analysis, but simplified loads are required for design.

From the above discussion, it is seen that the parameters of the dynamic system are subject to uncertainty ~~in their~~ evaluation. The analysis and design of a structure based on its equivalent dynamic system can only be carried out approximately. This is especially true in design since idealized resistances and simplified loads are used. Although rules are given at appropriate places which will help the designer to make reasonable assumptions and approximations, a basic understanding of the effect of dynamic loads is necessary for the intelligent application of these rules to actual problems.

In paragraphs 5-06 through 5-08, methods of analysis of single-degree dynamic systems are discussed. The results of a systematic analysis of single-degree dynamic systems for several simplified loadings are plotted in non-dimensional design charts in paragraph 5-10. The application of these charts to the design of a single-degree dynamic system is discussed in paragraphs 5-12 through 5-14. The analysis and design of multi-degree dynamic systems are discussed in the remaining portion of the manual.

ANALYSIS OF SINGLE-DEGREE-OF-FREEDOM SYSTEMS

5-06 INTRODUCTION. The analysis of single-degree dynamic systems consists of the evaluation of the displacement using Newton's equation of motion (equation 5.3). Two methods are generally used in solving this type of differential equation. One is the rigorous method in which the solution is obtained directly from the differential equation. This method is practical only when both the load and the resistance function can be expressed in simple mathematical forms. The other method is the numerical method in which the solution is obtained by a step-by-step integration. The numerical method is generally applicable for any type of load and resistance function. The two methods of analysis are discussed in paragraphs 5-07 and 5-08 respectively. A numerical example is given in paragraph 5-09 to illustrate the different methods of analysis.

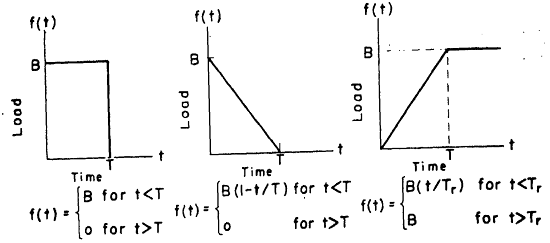
5-07 ANALYSIS BY RIGOROUS METHOD. a. General. The problem considered in this paragraph is the evaluation of the displacement using Newton's equation of motion for a single-degree dynamic system as shown in figure 5.1. The equation of motion is:

$$m \frac{d^2x}{dt^2} = f(t) - R \tag{5.15}$$

For a given load and a given resistance function, the procedure for evaluating the displacement is the same as that for solving a second order differential equation. First, the complementary function of the differential equation is obtained by assuming the load to be zero. There are two constants of integration in the complementary function. Next the particular integral for the given load is evaluated. The complete solution for the displacement is the sum of the complementary function and particular integral which are defined respectively as the transient and the forced solutions in dynamic problems [2, 3, 7]. From the known initial conditions, that is, the displacement and the velocity of the mass at the initiation of load, the constants of integration are evaluated. Thus an expression of the displacement as a function of time is obtained. It is obvious that the displacement function depends on the mass, m, the dynamic load, $f(t)$, and the resistance function, R.

The rigorous method of analysis is generally impractical for problems when either the dynamic load or the resistance is a complicated function. However, many important concepts related to the basic principle of dynamic analysis and design can be illustrated when this method is applied to idealized cases in which both the load and the resistance are of simple mathematical form.

Three types of dynamic load of simple form are shown in figure 5.6 together with the expressions for them. Each type of load is completely specified by two parameters; the peak load B and the load duration T .



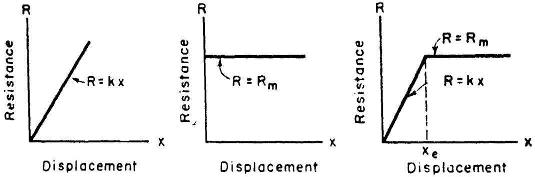
(a) Rectangular Load

(b) Triangular Load

(c) Step Load with Finite Rise Time, T_r

Figure 5.6. Three idealized dynamic loads

Three types of idealized resistance functions as shown in figure 5.7 are chosen for study. The dynamic systems characterized by these resistance



(a) Elastic

(b) Completely Plastic

(c) Elasto-Plastic

Figure 5.7. Three types of resistance function

functions are defined as: (a) linearly elastic, (b) completely plastic, and (c) elasto-plastic. The expressions of these resistance functions are also given in figure 5.7. It is noted that the linearly elastic and completely plastic systems are only special cases of the elasto-plastic system. The rigorous method of analysis of these basic types of single-degree dynamic systems are given separately in the following three paragraphs.

b. Linearly Elastic System. The resistance function of a linearly elastic system is always equal to kx . Substituting kx for R in equation (5.15), the equation of motion becomes:

$$m \frac{d^2x}{dt^2} = f(t) - kx \tag{5.16}$$

Referring to a textbook on Differential Equations, the complete solution of equation (5.16) is given by:

$$x = C \sin\left(2\pi \frac{t}{T_n} - \phi\right) + x_f \tag{5.17}$$

where

$T_n = 2\pi\sqrt{m/k}$ = natural period of oscillation

x_f = forced solution

C = constant of integration

ϕ = phase angle (constant of integration)

The sinusoidal term of equation (5.17) is the transient solution which is associated with the oscillatory nature of the dynamic system. The two constants of integration, C and ϕ , are determined from the requirement that the displacement and velocity of the mass must satisfy certain prescribed initial conditions. For structures under dynamic load, the usual initial conditions are:

$$\text{at } t = 0,\ x = 0, \text{ and } dx/dt = 0 \tag{5.18}$$

Once C and ϕ are determined, the displacement at any time is obtained.

Equation (5.17) shows that, for any given dynamic load, the speed of response of the linearly elastic system depends on the natural period of oscillation, T_n , in the transient solution. On the other hand, the rapidity of variation of the applied load depends on the load duration, T

Thus the non-dimensional ratio, T/T_n , is expected to be an important parameter in dynamic analysis and design.

As an illustrative example, the case of a triangular external load given in figure 5.6b is considered. The expressions for the load are different depending on whether t is greater or smaller than T . The corresponding displacement functions are also different.

For $t < T$, the expression for the load is:

$$f(t) = B\left(1 - \frac{t}{T}\right) \tag{5.19}$$

The forced solution for this load is:

$$x_f = \frac{B}{k}\left(1 - \frac{t}{T}\right) \tag{5.20}$$

Substituting equation (5.20) into equation (5.17) and taking the initial conditions given by equation (5.18) into account, the arbitrary constants C and ϕ can be evaluated. These are given by:

$$C = \frac{B}{k}\sqrt{1 + \left(\frac{T_n}{2\pi T}\right)^2} \tag{5.21}$$

$$\phi = \tan^{-1}\left(2\pi\frac{T}{T_n}\right) \tag{5.22}$$

Thus, in the time interval from 0 to T , the displacement is given by:

$$\frac{x}{x_s} = \sqrt{1 + \left(\frac{T_n}{2\pi T}\right)^2}\ \sin\left(2\pi\frac{T}{T_n}\frac{t}{T} - \phi\right) + \left(1 - \frac{t}{T}\right) \tag{5.23}$$

where

$x_s = \frac{B}{k}$ = static deflection produced by the peak load, B .

After $t = T$, the external load is zero. The forced solution is also zero. From the displacement and velocity at time, T , determined from equation (5.23), the constants of integration of the transient solution after $t > T$ can be determined. The displacement function after $t > T$ is given by:

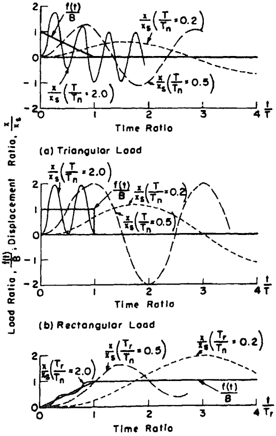
$$\frac{x}{x_s} = C_1 \sin\left[2\pi\left(\frac{T}{T_n}\right)\left(\frac{t}{T}\right) - \phi_1\right] \tag{5.24}$$

where

$$C_1 = \left[1 + \frac{1}{2}\left(\frac{T_n}{\pi T}\right)^2 - \frac{T_n}{\pi T}\sqrt{1 + \left(\frac{T_n}{2\pi T}\right)^2}\cos\left(2\pi\frac{T}{T_n} - \phi\right)\right]^{\frac{1}{2}}$$

$$\phi_1 = 2\pi\left(\frac{T}{T_n}\right) - \tan^{-1}\left\{\frac{\sqrt{\left(\frac{2\pi T}{T_n}\right)^2 + 1}\left[\sin\left(2\pi\frac{T}{T_n} - \phi\right)\right]}{-1 + \sqrt{\left(\frac{2\pi T}{T_n}\right)^2 + 1}\left[\cos\left(2\pi\frac{T}{T_n} - \phi\right)\right]}\right\}$$

Equations (5.23) and (5.24) show that the non-dimensional displacement variable x/x_s at a given time, depends only on T/T_n. In figure 5.8a, the displacement of a linearly elastic system subjected to a triangular load is plotted versus time for three values of T/T_n, namely, $T/T_n = 0.2$, 0.5, and 2.0. The corresponding cases of rectangular load and step load with rise time, T_r, are plotted in figures 5.8b and c. These curves show: (1) the maximum displacement, (2) the time when the maximum displacement occurs, and (3) that the time for one complete oscillation is equal to T_n, and (4) the displacement at any time.



(a) Triangular Load

(b) Rectangular Load

(c) Step Load with Rise Time T_r

Figure 5.8. The displacement functions of elastic systems subjected to three types of external load

c. Completely Plastic System. The resistance of a completely plastic system is a constant equal to R_m. No practical system ever behaves like a plastic system. But for a given system, if the maximum displacement x_m is much larger than the maximum elastic deflection x_e, the elastic portion of the resistance function can be neglected and the system treated as a completely plastic system.

Substituting R_m for R in equation (5.15), the equation of motion of a completely plastic system is:

$$m \frac{d^2x}{dt^2} = f(t) - R_m \tag{5.25}$$

Once $f(t)$ is given, the displacement function can be evaluated directly by integration.

For the case of a rectangular load given in figure 5.6a, the expressions of the load are:

$$f(t) = B \quad \text{for } t < T \tag{5.26}$$

$$f(t) = 0 \quad \text{for } t > T \tag{5.27}$$

Substituting equations (5.26) and (5.27) into equation (5.25), integrating and taking equation (5.18) into account, the displacement functions of the mass for a rectangular load are given by:

$$\text{for } t < T,\ \frac{x}{x_s'} = \left(1 - \frac{R_m}{B}\right)\left(\frac{t}{T}\right)^2 \tag{5.28}$$

$$\text{for } T < t < t_m,\ \frac{x}{x_s'} = \left(1 - \frac{R_m}{B}\right) + 2\left(1 - \frac{R_m}{B}\right)\left(\frac{t}{T} - 1\right) - \frac{R_m}{B}\left(\frac{t}{T} - 1\right)^2 \tag{5.29}$$

where

$$x_s' = \frac{1}{2}\frac{BT^2}{m}$$

and

t_m = the time when the maximum displacement x_m is reached.

Substituting equation (5.28) into equation (5.6) and integrating, the work done by the external load up to any time, t, which is less than T can be obtained. This is given by equation (5.30).

$$\frac{W}{W_p} = \left(1 - \frac{R_m}{B}\right)\left(\frac{t}{T}\right)^2 \tag{5.30}$$

where

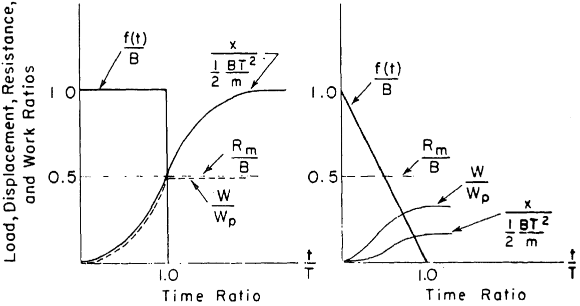
$$W_p = \frac{1}{2}\frac{H^2}{m} \text{ (see equation 5.12)}$$

$$= \frac{1}{2}\frac{(BT)^2}{m}$$

The maximum work done by the load occurs at time, T , and is given by:

$$\frac{W_m}{W_p} = \frac{1}{2}\left(1 - \frac{R_m}{B}\right) \tag{5.31}$$

The preceding four equations show that the non-dimensional quantity R_m/B is an important parameter in the analysis and design of completely plastic systems. In figure 5.9a, the displacement and the work done by the load are plotted versus time for a rectangular load with $B = 2R_m$. The corresponding functions for a triangular load are plotted in figure 5.9b. These curves show: (1) the displacement function for a completely plastic system is non-oscillatory; (2) the maximum displacement; (3) the maximum work done; and (4) the time t_m when the maximum displacement x_m occurs.



(a) Rectangular Load (b) Triangular Load

Figure 5.9. The displacement and work done on a completely plastic system $B = 2R_m$

d. Elasto-Plastic System. This is the dynamic system which is commonly used to represent a structure as a whole, as well as its members, for the purpose of analysis and design of blast resistant structures. Under blast loads, it is often uneconomical to design a structure to behave entirely in the elastic range. Many ductile construction materials are characterized by a large yield range which enables them to absorb a large amount of energy. Use of this property in design results in much lighter members than would be required should the energy absorption ability of the yielding region not be utilized.

For an elasto-plastic system, the resistance function is given by:

$$R = kx \qquad \text{for } x < x_e \tag{5.32a}$$

$$R = R_m \qquad \text{for } x > x_e \tag{5.32b}$$

After the maximum displacement is reached, the dynamic system rebounds and the resistance function is given by:

$$R = k\left[x_e - \left(x_m - x\right)\right] \qquad \text{for } \left(x_m - x\right) < 2x_e \tag{5.32c}$$

$$R = - R_m \qquad \text{for } \left(x_m - x\right) > 2x_e \tag{5.32d}$$

For the rectangular and triangular loads shown in figure 5.6, the expressions of the load are different depending on whether t is greater or smaller than T . Hence, the six differential equations given below are needed to express the equation of motion.

$$t < T,\ x < x_e,\ m\left(\frac{d^2x}{dt^2}\right) = f(t) - kx \tag{5.33a}$$

$$t < T,\ x > x_e,\ m\left(\frac{d^2x}{dt^2}\right) = f(t) - R_m \tag{5.33b}$$

$$t > T,\ x < x_e,\ m\left(\frac{d^2x}{dt^2}\right) = - kx \tag{5.33c}$$

$$t > T,\ x > x_e,\ m\left(\frac{d^2x}{dt^2}\right) = - R_m \tag{5.33d}$$

$$T > t > t_m,\ m\,\frac{d^2x}{dt^2} = f(t) - k\left[x_e - \left(x_m - x\right)\right] \tag{5.33e}$$

$$t > t_m \text{ and } t > T,\ m\,\frac{d^2x}{dt^2} = - k\left[x_e - \left(x_m - x\right)\right] \tag{5.33f}$$

The first four equations express the motion before the maximum displacement is reached and the last two equations express the motion when the dynamic system is in elastic rebound.

The sequence of using these equations for the evaluation of the displacement function of a given problem depends on: (1) the time, t_e, when the maximum elastic deflection is reached, and (2) the time, t_m, when the maximum displacement is reached.

For the linearly elastic system, T/T_n is an important parameter while R_m/B is important for a plastic system. Both parameters are important for an elasto-plastic system. As an example, the case of a triangular load as shown in figure 5.6b is considered. The numerical values of the non-dimensional parameters are assumed for illustrative purposes as:

$$C_T = \frac{T}{T_n} = 0.3$$

$$C_R = \frac{R_m}{B} = 0.5$$

Equation (5.33a) is first used. Following the method described in paragraph 5-07b, the displacement function is given by:

$$x = \frac{B}{k}\left[\sqrt{1 + \left(\frac{T_n}{2\pi T}\right)^2}\,\sin\left(2\pi \frac{T}{T_n}\frac{t}{T} - \phi\right) + \left(1 - \frac{t}{T}\right)\right] \tag{5.34}$$

where

$$\phi = \tan^{-1} 2\pi \frac{T}{T_n} = 62^{\circ}4'$$

Equation (5.34) is true so long as $t < T$ and $x < x_e$. Substituting

$$x = x_e = \frac{R_m}{k} = C_R \frac{B}{k}$$

into equation (5.34), the time, t_e, when the maximum elastic deflection, x_e, is reached can be evaluated. This is given by:

$$t_e/T = 0.64$$

The displacement and velocity of the mass at time, t_e, are respectively given by:

$$x\left(t_e\right) = x_e = C_R \frac{B}{k}$$

$$v\left(t_e\right) = 1.11 \frac{B}{kT}$$

In the time interval, $t_e < t < T$, equation (5.33b) should be used. The displacement function is obtained by direct integration. This is given by:

$$x = \frac{B}{k}\left\{ C_R + 1.11\left(\frac{t}{T} - \frac{t_e}{T}\right) + \frac{1}{2}\left(\frac{2\pi T}{T_n}\right)^2 \left(1 - C_R\right)\left(\frac{t}{T} - \frac{t_e}{T}\right) - \frac{1}{6}\left(\frac{2\pi T}{T_n}\right)^2 \left[\left(\frac{t}{T}\right)^3 - 3\,\frac{t}{T}\left(\frac{t_e}{T}\right)^2 + 2\left(\frac{t_e}{T}\right)^3\right]\right\} \qquad (5.35)$$

At time, $t = T$, the displacement and velocity are respectively given by:

$$x(T) = 0.831\,\frac{B}{k}$$

$$v(T) = 0.702\left(\frac{B}{kT}\right)$$

In the time interval, $T < t < t_m$, equation (5.33d) should be used. The displacement in this time interval is:

$$x = \frac{B}{k}\left[0.831 + 0.702\left(\frac{t}{T} - 1\right) - \frac{C_R}{2}\left(\frac{2\pi T_n}{T}\right)^2\left(\frac{t}{T} - 1\right)^2\right] \qquad (5.36)$$

From equation (5.36), the maximum displacement, x_m, is given by:

$$x_m = 0.971\,\frac{B}{k}$$

and the maximum displacement occurs at

$$\frac{t_m}{T} = 1.397$$

After $t > t_m$, the dynamic system is in elastic rebound, and equation (5.33f) should be used. The displacement after $t = t_m$ is given by:

$$x = \frac{B}{k}\left[0.471 + \frac{1}{2}\cos\left(2\pi\,\frac{t - t_m}{T}\right)\right] \qquad (5.37)$$

The displacement function as expressed by equations (5.34), (5.35), (5.36), or (5.37) is plotted in figure 5.10 in non-dimensional form.

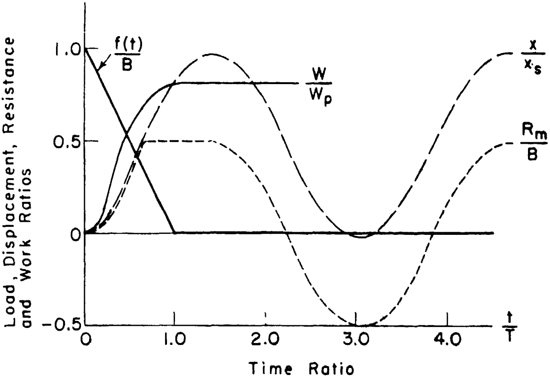


Figure 5.10. Displacement and work done by a triangular load on an elasto-plastic system

$$C_T = \frac{T}{T_n} = 0.3, \quad C_R = \frac{R_m}{B} = 0.5$$

Once the displacement function is known, the work done by the external load at any time, as expressed by equation (5.6), can be evaluated and this is also plotted in figure 5.10. The maximum work done by the external load, which is equal to the total strain energy at the maximum displacement, is given by equation (5.38).

$$W_m = \int_0^{t_m} f(t)\left(\frac{dx}{dt}\right)dt \doteq R_m\left(x_m - \frac{x_e}{2}\right) \qquad (5.38)$$

As shown in figure 5.10, the ratio of W_m/W_p for the given problem is equal to 0.814.

5-08 ANALYSIS BY NUMERICAL METHODS. a. Need for Numerical Method. The rigorous method described in paragraph 5-07 is practical only for the cases

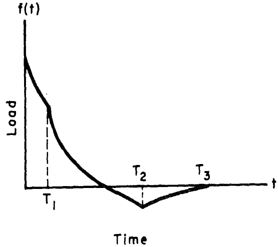
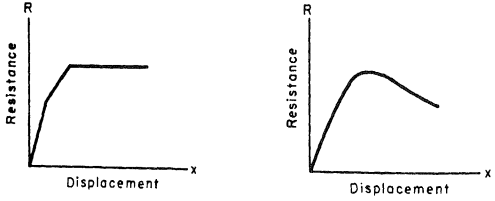


Figure 5.11. A typical dynamic load

when both the load and the resistance are simple mathematical expressions. However, for an actual problem, the load may be quite complicated. A typical case is shown in figure 5.11. Four separate equation are needed to express f(t) . For this load applied to an elastoplastic system, the equations of motion are expressed by eight separate differential equations. The solutio of these equations by rigorous metho is generally impractical.

The rigorous method is further complicated if the resistance function is not a simple shape as shown in figure 5.7. Two practical cases are shown in figure 5.12. One is the idealized resistance function of a fixed-end beam, and the other is of the columns of a single story frame building when the vertical load is taken into consideration in the evaluation of the resistance.

For practical applications, when the load and resistance functions are complex, because of the impracticability of solving the rigorous



(a) Resistance Function of a Fixed-end Beam

(b) Resistance Function of a One-Story Building

Figure 5.12. Two practical resistance functions

differential equations, it is necessary to have other methods of analysis. Many methods have been developed. One is the method of numerical integration which is discussed in this section.

Perhaps it should be mentioned that these methods are by no means inexact. Any degree of accuracy desired in the end result can be obtained. But, for the convenience of computation, these methods always embody simplifying assumptions which affect the accuracy of the results. The inaccuracies are not due to inherent properties of the numerical methods, but rather due to the simplifying assumptions used in carrying them out.

b. Basic Principles of Numerical Analysis. The differential equation of motion (equation 5.15) expresses implicitly the acceleration of a dynamic system at any time, $a = \frac{f(t) - R}{m}$. It is known that the velocity and displacement can be obtained from the acceleration by integration. The numerical method of evaluating the displacement from equation (5.15) is called the method of numerical integration.

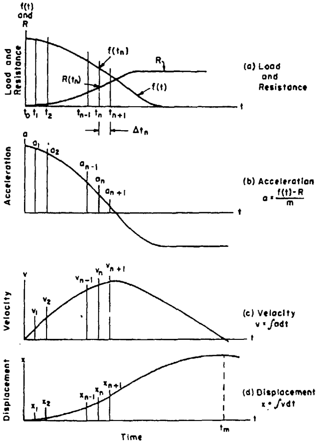


Figure 5.13. Load, resistance, acceleration, velocity and displacement versus time

The load, resistance, acceleration, velocity, and displacement are plotted versus time in figure 5.13 for a typical case.

Suppose t_o, t_1, $t_2 \ldots t_{n-1}$, t_n, $t_{n+1} \ldots$ is a time sequence. The time interval from t_n to t_{n+1} is denoted by Δt_n . The dynamic load is assumed to be initiated at $t = t_o$. The acceleration, velocity, and displacement at t_n are denoted by a_n, v_n, and x_n, respectively. If the acceleration in the time interval, Δt_n , is represented by $a(t)$, the velocity and displacement at t_{n+1} are given respectively by the following equations:

$$v_{n+1} = v_n + \int_{t_n}^{t_{n+1}} a(t)\, dt \tag{5.39}$$

$$x_{n+1} = x_n + v_n \left(\Delta t_n\right) + \int_{t_n}^{t_{n+1}} \left[\int a(t)\, dt\right] dt \tag{5.40}$$

These two equations indicate that the velocity and displacement at t_{n+1} can be obtained by extrapolation from the corresponding values at t_n once the acceleration in the time interval, Δt_n , is known.

In the analysis of structures under dynamic load, the velocity and displacement at the initiation of load are given as zero, that is:

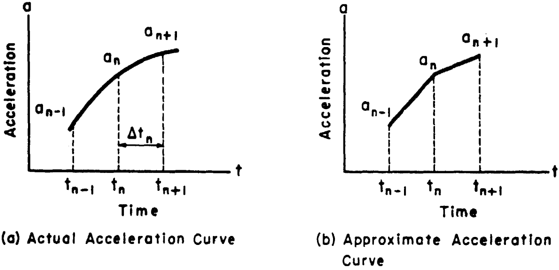
$$v_o = 0 \text{ and } x_o = 0 \tag{5.41}$$

In applying equations (5.39) and (5.40), the values of v_1 and x_1 can be obtained provided that $a(t)$ is known in the time interval from t_o to t_1 . From the values of v_1 and x_1 , the values of v_2 and x_2 can be obtained. This process can be continued until the values of v_n and x_n for any values of n are obtained. Hence, in the above numerical procedure the displacement, x_1, x_2, $x_3 \ldots$, $x_n \ldots$ are evaluated by a step-by-step extrapolation method starting from the known initial conditions.

Here the difference between the rigorous method and the numerical method can be perceived. In the rigorous method, the value of x is obtained as a continuous function of time while in the numerical method, the value of x is obtained at discrete values of time. In other words, the actual relationship between x and t is given by a differential equation, but in the numerical method, this relationship is replaced by a finite difference equation. As the time interval in the sequence of time is reduced, the finite difference equation approaches the actual differential equation as a limit. It can, therefore, be concluded that the accuracy of the numerical method based on the finite difference equation for most problems depends almost entirely on the magnitude of the successive time intervals.

Many extrapolation formulas have been derived for the solution of the finite difference equation [5]. For engineering application, simplified extrapolation formulas are often used which are obtained by assuming a simple acceleration-time relationship in any time interval, Δt_n. Two procedures are developed in paragraphs 5-08c and 5-08d. In one case the acceleration is assumed to be varying linearly with time. In the other case, the acceleration is assumed to consist of a series of pure pulses.

c. Linear Acceleration Extrapolation Method. In this method, the acceleration, $a(t)$, in any time interval, Δt_n, as shown in figure 5.14a, is assumed to vary linearly with time as shown in figure 5.14b.



(a) Actual Acceleration Curve

(b) Approximate Acceleration Curve

Figure 5.14. Comparison of actual and approximate acceleration curve used in linear acceleration extrapolation method

In the time interval, Δt_n, $a(t)$ is then given by:

$$a(t) = a_n + \left(\frac{a_{n+1} - a_n}{\Delta t_n}\right)\left(t - t_n\right) \tag{5.42}$$

where $\Delta t_n = t_{n+1} - t_n$

Substituting equation (5.42) into equations (5.39) and (5.40), the recurrence formulas for velocity and displacement are given respectively by:

$$v_{n+1} = v_n + \left[\left(\frac{a_n + a_{n+1}}{2}\right)\right] \Delta t_n \tag{5.43}$$

$$x_{n+1} = x_n + v_n\,(\Delta t) + \left(\frac{a_n}{3} + \frac{a_{n+1}}{6}\right)\left(\Delta t_n\right)^2 \tag{5.44}$$

Once a_n and a_{n+1} are known, these two equations are applicable to the evaluation of the velocity and displacement at t_{n+1} from the known values at t_n

According to definition, $a(t)$ is given by:

$$a(t) = \frac{d^2x}{dt^2} = \frac{1}{m}\left[f(t) - R\right]$$

and a_{n+1} is:

$$a_{n+1} = \frac{1}{m}\left[f\left(t_{n+1}\right) - R\left(t_{n+1}\right)\right] \tag{5.45}$$

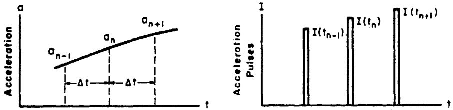
In equation (5.45), $f\left(t_{n+1}\right)$ is known, but the resistance, $R\left(t_{n+1}\right)$, may depend on x_{n+1}, which is still an unknown. Hence, in the application of equations (5.43) and (5.44), a trial and error procedure has to be adopted. A value of x_{n+1} is assumed [as a first trial, x_{n+1} can be assumed to be $x_n + v_n\left(\Delta t_n\right)$] from which $R\left(t_{n+1}\right)$ and a_{n+1} are obtained. The value of x_{n+1} is computed from equation (5.44) and the computed value is compared with the assumed value. This procedure is repeated until the assumed and computed values of x_{n+1} agree with each other.

Of course, in the plastic region, the resistance is a constant; x_{n+1} can be obtained directly from equation (5.44) and the trial and error procedure is not necessary.

There are two basic characteristics of this method. First, it is, in general, a trial and error method, hence it may be inconvenient to carry out, and secondly, the acceleration is assumed to vary linearly with time, thus the variation of acceleration in the time interval, Δt_n, is partially taken into account. So this method is called a trial and error linear acceleration extrapolation method.

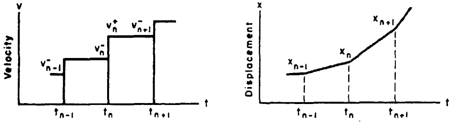
In the application of the above method, the successive time intervals, Δt_0, Δt_1, $\Delta t_2, \ldots \Delta t_n$ do not have to be equal although they are chosen to be equal for the convenience of computation. To determine the magnitude of the time interval to be used for a given problem, a number of factors have to be considered. First, the time interval must be chosen such that within it the acceleration can be reasonably approximated by a straight line. Secondly, the convergence of the trial and error procedure must be rapid. Thirdly, the number of steps necessary for the evaluation of the maximum displacement must be reasonable. For the first and second considerations, the time interval should be small while for the third consideration, the time interval should be as large as tolerable. Because of these conflicting requirements, it is found that a time interval equal to about $T_n/6$ is a good compromise to be used in the elastic region, but a larger time interval can be used in the plastic region provided that the external load in the time interval is a reasonably straight line.

d. Acceleration Impulse Extrapolation Method. In this method, the actual acceleration curve shown in figure 5.15a is replaced by a train of



(a) Given Acceleration Curve

(b) Acceleration Pulses to Replace the Given Acceleration Curve



(c) Velocity

(d) Displacement

Figure 5.15. Acceleration impulse extrapolation method

equally spaced impulses occurring at t_o, t_1, t_2, ... t_n. The magnitude of the acceleration impulse at t_n is given by:

$$I\left(t_n\right) = a_n\,(\Delta t) \qquad (5.46)$$

where

$$\Delta t = t_1 - t_o = t_2 - t_1 = t_3 - t_2 = \ldots\ldots = t_n - t_{n-1}$$

This is shown in figure 5.15b. Since an impulse is applied at t_n, there is a discontinuity in the value of velocity at t_n. In the time interval from t_n to t_{n+1} the velocity is constant and the displacement varies linearly with time. The velocity and displacement thus obtained are shown in figures 5.15c and 5.15d.

Suppose t_n^- and t_n^+ indicate the time immediately before and after the application of the impulse at t_n, and let v_n^- and v_n^+ indicate respectively the velocity at t_n^- and t_n^+; these two velocities are related by the following equation:

$$v_n^+ = v_n^- + a_n\,(\Delta t) \qquad (5.47)$$

The relationship between x_{n-1} and x_n, and between x_n and x_{n+1} are given by:

$$\begin{aligned} x_n - x_{n-1} &= v_n^-\,(\Delta t) \\ x_{n+1} - x_n &= v_n^+\,(\Delta t) \end{aligned} \qquad (5.48)$$

Combining equations (5.47) and (5.48), the three successive displacements are related by:

$$x_{n+1} = 2x_n - x_{n-1} + a_n\,(\Delta t)^2 \qquad (5.49)$$

This is the basic recurrence formula for the acceleration impulse extrapolation method. Once the values of x at t_{n-1} and t_n are known, the value at t_{n+1} can be directly computed without resorting to a trial and error procedure.

The evaluation of x_1 by the recurrence formula needs special consideration. In equation (5.49), when $n = 0$, x_1 is given by:

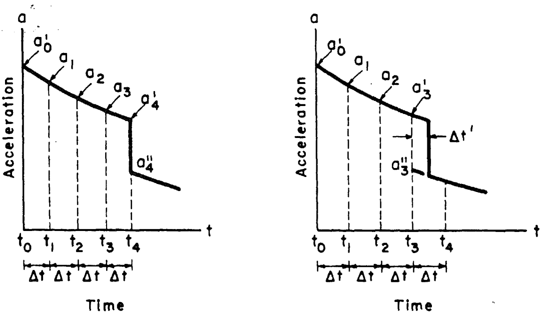
$$x_1 = 2x_o - x_{-1} + a_o \, (\Delta t)^2$$

Both x_o and x_{-1} are equal to zero, and the expression of x_1 is simplified to:

$$x_1 = a_o \, (\Delta t)^2 \tag{5.50}$$

The correct value of a_o used in the computation is not equal to a_o'' as indicated in figure 5.16, curve a or curve b, but is given by:

$$a_o = \frac{1}{2} a_o' + \frac{1}{6} \left(a_1 - a_o'\right) \tag{5.51}$$



(a) (b)

Figure 5.16. Discontinuities in the acceleration curve

When equation (5.51) is used in equation (5.50) for the evaluation of x_1, the value thus obtained is that given by equation (5.44) which is based on the Linear Extrapolation Method. In order to avoid a trial and error procedure for the evaluation of x_1, the value of a_1 in equation (5.51) is taken to be $\frac{f(t_1)}{m}$ instead of its exact value $\frac{f(t_1) - R(t_1)}{m}$. In figure 5.16a, if $a_o' = 0$, a_o is equal to $\frac{a_1}{6}$. In figure 5.16b if a_o' and a_1 are approximately equal to each other, the value $\frac{a_o'}{2}$ may be used for a_o Equation (5.49) is directly applicable for the evaluation of x_2, $x_3 \ldots x_{n+1}$ whenever the acceleration, $a(t)$, is a continuous curve. When there is a discontinuity in the acceleration this equation is still applicable provided a modified value of the acceleration is used. This is illustrated in figures 5.16a and b. In figure 5.16a, there is a discontinuity at t_4 which occurs at the end of the time interval, Δt. Under this condition, the value of a_n used in the numerical procedure is the average value at discontinuity, namely $\frac{1}{2}\left(a_4' + a_4''\right)$ for a_4. In figure 5.16b, the discontinuity occurs within the time interval, Δt. The correct value for a_3 in this procedure is:

$$a_3 = \frac{1}{2}\left[a_3 + a_3'' + \left(a_3' - a_3''\right)\left(\frac{\Delta t'}{\Delta t}\right)^2\right] \qquad (5.52)$$

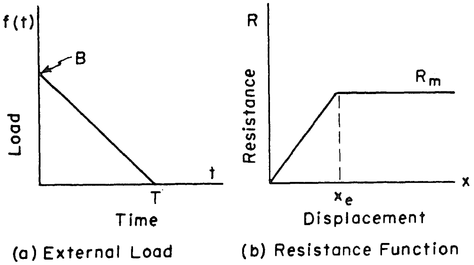
For a given problem if there are one or two discontinuities, the method described above can be applied conveniently. However, if the number of discontinuities is large, it is more convenient to replace the given acceleration by a smooth curve.

The main advantage of the acceleration impulse extrapolation method is that the numerical computation is straightforward and involves no trial and error procedures. Its main disadvantage is that the result is less accurate than the linear acceleration method. It has been proved that if the acceleration curve is a straight line or a series of straight lines with the same slope, the result obtained from the acceleration impulse method is exact. However, the actual acceleration curve may be rather complicated so that the result obtained by this method is usually only approximate. A number of examples have been carried out, and the results indicate that the accuracy in the evaluation of the maximum displacement is within 3% provided the time interval used in the computation is not greater than $T_n/10$.

Another disadvantage of this method is that any numerical mistake in the computation is increased by geometrical progression in subsequent steps. Hence, one must be extremely careful to avoid any numerical mistake especially in the early stages of computation.

5-09 NUMERICAL EXAMPLE. An example is given to illustrate the actual application of the different methods of analysis. The same example is evaluated by three different methods: (1) the rigorous method; (2) the linear acceleration extrapolation method; and (3) the acceleration impulse extrapolation method.

The external load and the resistance function for the example are shown in figures 5.17a and 5.17b respectively.



(a) External Load (b) Resistance Function

Figure 5.17. External load and resistance function for the illustrative example

The assumed dimensional parameters of the dynamic system are

$m = 2.5$ kips-sec^2/ft
$k = 9860$ kips/ft
$R_m = 750$ kips
$x_e = 0.0760$ ft
$T = 0.10$ sec

$$B = 1000 \text{ kips}$$

$$T_n = 2\pi\sqrt{\frac{m}{k}} = 0.10 \text{ sec}$$

The non-dimensional parameters are:

$$C_R = \frac{R_m}{B} = 0.75, \qquad C_T = \frac{T}{T_n} = 1.00$$

In the rigorous method, the numerical values of the parameters are substituted into equation (5.23). The displacement in the time interval, $0 < t < t_e$ is given by:

$$x = 0.01012 \left[1.012 \sin \left(2\pi \frac{t}{T_n} - 80^\circ 57' \right) + \left(1 - \frac{t}{T} \right) \right] \tag{5.53a}$$

The time, t_e , when the displacement x is equal to the maximum elastic deflection, x_e , can be solved from equation (5.53a) and is equal to 0.022 sec. The displacement and velocity of the mass at t_e are:

$$x\left(t_e\right) = x_e = 0.0760 \text{ ft}$$

$$v\left(t_e\right) = 0.01012 \; (1.012) \left(\frac{2\pi}{T_n}\right) \cos \left(2\pi \frac{t_e}{T_n} - 80^\circ 57' \right) = 5.44 \text{ ft/sec}$$

Substituting the numerical values of the parameters into equation (5.33b), and integrating, the displacement function in the time interval, $t_e < t < t_m$, is given by:

$$x = 0.0760 + 5.44\,(t - 0.022) + 6(t - 0.022)^2 - 666(t - 0.022)^3 \tag{5.53b}$$

From equation (5.53b), it is found that the maximum displacement is

$$x_m = 0.2825 \text{ ft}$$

which occurs at t_m given by

$$t_m = 0.077 \text{ sec}$$

The displacement computed from either equation (5.53a) or equation (5.53b) is tabulated in column 4 of table 5.1.

The same example is evaluated by applying the two methods of numerical analysis described in paragraph 5-08. The detailed computations of the linear acceleration and acceleration impulse extrapolation methods are given in tables 5.2 and 5.3 respectively. The displacements obtained by these three methods are tabulated in table 5.1. It is seen that the error in using the numerical methods in the evaluation of the maximum displacement is less than 1.5%.

Table 5.1. Comparison of the Results Obtained by Different Methods of Analysis

Time	Displacement Function x		
t (sec)	Linear Acceleration Extrapolation Method	Acceleration Impulse Extrapolation Method	Rigorous Method
0	0	0	0
0.01	0.0181	0.0200	0.0186
0.02	0.0633	0.0683	0.0651
0.03	0.1182	0.1216	0.1203
0.04	0.1713	0.1729	0.1718
0.05	0.2184	0.2182	0.2184
0.06	0.2556	0.2535	0.2548
0.07	0.2787	0.2748	0.2785
0.08	0.2839	0.2781	
			t_m = 0.077 sec
0.09		0.2594	x_m = 0.2825
Maximum Displacement	0.2839	0.2781	0.2825

Table 5.2. Details of Computation by the Linear Acceleration Extrapolation Method

n	Trial No.	t_n (sec)	x_n (ft)	v_n (ft/sec)	$a_n = (P_n - R_n)/m$ (ft/sec^2)	Assumed x_{n+1} (ft)	P_{n+1} (kips)	R_{n+1} (kips)	$a_{n+1} = (P_{n+1} - R_{n+1})/m$ (ft/sec^2)	v_{n+1} (ft/sec)	Computed x_{n+1} (ft)
0	1st	0	0	0	+400	0.0150	900	145	+282	3.41	0.0180
	2nd					0.0180	900	177	+289	3.44	0.0181
	3rd					0.0181	900	180	+288	3.44	0.0181
1	1st	0.01	0.0181	+3.44	+288	0.0525	800	518	+113	5.44	0.0640
	2nd					0.0640	800	631	+68	5.22	0.0632
	3rd					0.0633	800	624	+70	5.23	0.0633
2		0.02	0.0632	+5.23	+70	*	700	750	-20	5.48	0.1182
3		0.03	0.1182	+5.48	-20	*	600	750	-60	+5.08	0.1713
4		0.04	0.1713	+5.08	-60	*	500	750	-100	+4.28	0.2184
5		0.05	0.2184	+4.28	-100	*	400	750	-140	+3.08	0.2556
6		0.06	0.2556	+3.08	-140	*	300	750	-180	+1.48	0.2787
7		0.07	0.2787	-1.48	-180	*	200	750	-220	-0.52	0.2839

* The displacement is larger than the maximum elastic deflection. R_{n+1} is a constant equal to 750 kips. The computation of v_{n+1} and x_{n+1} becomes straightforward without resorting to a trial and error procedure.

Note: $P_n = f(t_n) = 1000\left[1 - \frac{n(\Delta t)}{0.1}\right]$; $t = 0.01$ sec

$R_n = kx_n = 9860\, x_n$ for $x < x_e = 0.0760$ ft; $R_n = R_m = 750$ kips for $x_n > 0.0760$ ft

$$v_{n+1} = v_n + \left(\frac{a_n + a_{n+1}}{2}\right)(\Delta t), \quad x_{n+1} = x_n + v_n(\Delta t) + \left(\frac{a_n}{3} + \frac{a_{n+1}}{6}\right)(\Delta t)^2$$

Table 5.3. Details of Computation by Acceleration Impulse Extrapolation Method

n	t (sec)	P_n (kips)	R_n (kips)	$P_n - R_n$ (kips)	$a_n = (P_n - R_n)/m$ ft/sec^2	$a_n(\Delta t)^2$ (ft)	$2x_n$ (ft)	x_{n-1} (ft)	x_{n+1} (ft)
0	0	+1000/2	0	+500	+200	+0.02	0	0	+0.0200
1	+0.01	+900	+197	+703	+283	+0.0283	+0.0400	0	+0.0683
2	+0.02	+800	+674	+126	+50.4	+0.0050	+0.1366	+0.0200	+0.1216
3	+0.03	+700	+750	-50	-20	-0.0020	+0.2432	+0.0683	+0.1729
4	+0.04	+600	+750	-150	-60	-0.0060	+0.3458	+0.1216	+0.2182
5	+0.05	+500	+750	-250	-100	-0.0100	+0.4364	+0.1729	+0.2535
6	+0.06	+400	+750	-350	-140	-0.0140	+0.5070	+0.2182	+0.2748
7	+0.07	+300	+750	-450	-180	-0.0180	+0.5496	+0.2535	+0.2781
8	+0.08	+200	+750	-550	-220	-0.0220	+0.5562	+0.2748	
9	+0.09								

Note: $P_n = f(t_n) = 1000\left[1 - n\left(\frac{\Delta t}{0.1}\right)\right]$

$$R_n = \begin{cases} kx_n & \text{for } x_n < x_e = 0.0760 \\ R_m = 750 & \text{for } x_n > x_e \end{cases}$$

$x_{n+1} = 2x_n - x_{n-1} + a_n(\Delta t)^2$

$(\Delta t) = 0.01$ sec

5-10 DESIGN CHARTS FOR SIMPLIFIED LOADINGS. a. General. Paragraph 5-10 presents, in non-dimensional graphs, the results of a systematic analysis of single-degree dynamic systems for simplified loads. In order to make these graphs simple and convenient for design use, only those cases which involve no more than three independent non-dimensional parameters are considered. For these cases, the complete results of analysis can be clearly presented on a simple graph with one parameter as the abscissa, a second parameter as the ordinate, and the third as a running parameter.

From the theory of dimensional analysis [1], the number of independent non-dimensional parameters needed to express the functional relationship of all the variables of a given problem is given by equation (5.54).

$$N = n - m \qquad (5.54)$$

where

N = number of non-dimensional parameters

n = number of dimensional parameters

m = number of fundamental dimensional quantities

For a single-degree dynamic system, there are three fundamental dimensional quantities: namely, mass, length, and time. If consideration is limited to problems with three or less non-dimensional parameters, the total number of dimensional variables can be no more than six. Of the six variables, one is the maximum displacement and one is the mass of the system. Hence, the load and the resistance must be completely specified by no more than four variables. If two variables, the spring constant, k , and the plastic resistance, R_m , are needed for the resistance function, the dynamic load must be completely specified by no more than two variables. These are the peak load, B , and the load duration, T. It is for this reason that only very simple types of loads as shown in figure 5.6 are chosen as the basis for the preparation of design charts. Although only three simplified loads are considered, most actual loads can be replaced by one of these with sufficient accuracy for preliminary design purposes.

The non-dimensional graphs for the linearly elastic, completely plastic and elasto-plastic systems subjected to the simplified loads shown in figure 5.6 are given separately in the following three sections. The non-dimensional parameters chosen for each case are also discussed. For the linearly elastic system, all three types of loads are used but only the rectangular and triangular loads are used for the completely plastic and elasto-plastic systems.

b. Linearly Elastic System. For a linearly elastic system subjected to simplified loads shown in figure 5.6, the problem involves five dimensional variables; x_m, m, k, B, and T Hence, only two non-dimensional parameters are needed in the presentation of the design charts and these parameters are chosen to be:

$$\text{D.L.F.} = \frac{x_m}{x_s} = \text{Dynamic Load Factor}$$

$$C_T = \frac{T}{T_n}$$

where x_s is equal to B/k which is the displacement produced in the system when the peak load B is applied statically.

The method described in paragraph 5-07b can be used for the evaluation of the dynamic load factor. The displacement function for a given load is derived, from which the maximum displacements, x_m , for different values of C_T and the time, t_m , at which x_m's occur, are obtained.

The dynamic load factors for the rectangular load, the triangular load and the step load with rise time, T_r , are plotted respectively in figures 5.18, 5.19, and 5.21. The time ratios t_m/T are also plotted in these figures for each loading. The composite of figures 5.18 and 5.19 is given in figure 5.20 for the purpose of comparing the effects of the rectangular and triangular loads.

The dynamic load factor in this manual is defined as $\frac{x_m}{x_s}$ where both x_m and x_s are the displacements of the equivalent dynamic system. In the evaluation of the equivalent dynamic system, it is assumed that the deflected shapes of the structure, for example a simply-supported beam, at any two instants of time are geometrically similar to each other. Under this assumption, the slope and curvature at a given point of the beam depend only on the mid-span deflection of the beam Thus the dynamic load factors also give the ratio of the maximum internal force developed at any point under dynamic load to the internal force developed at the same point when the peak load, B , is applied statically.

It should be noted that ordinary structures such as beams or frame buildings, involve a continuous distribution of mass. High modes of oscillation as well as the fundamental mode which is used in the evaluation of the single-degree dynamic system are present. As shown in appendix A, EM 1110-345-416, although the higher modes have very little effect as far as the mid-span deflection of beams is concerned, their effects on the bending moment and shear are quite large. Strictly speaking, the dynamic load factor based on a single-degree dynamic system, can only be referred to as the ratio of displacement but not as the ratio of internal forces. For the correct evaluation of internal forces such as bending moment and shear, high modes have to be considered and the structure must be replaced by a multi-degree dynamic system. In practical application, for the sake of simplicity a single-degree equivalent system is still generally used to approximate the given structure. The same dynamic load factor is applied to the displacement as well as to the bending moment. The high modes of oscillation are sometimes partially taken into account in the evaluation of the shear and this is discussed in appendix A, EM 1110-345-416.

Another point worth mentioning is that the dynamic load factor for the equivalent system of a structural member and the dynamic load factor for the equivalent system of a structure as a whole have slightly different meanings. For the equivalent system of a member, an elastic beam for example, the elastic strain energy is stored all along the beam. If the shape of deflection curve remains the same, the internal force at any point of the beam is proportional to the mid-span deflection. Hence, the dynamic load factor gives the ratio of internal forces under dynamic and static loads at any point of the beam. But for the equivalent system of a structure as a whole, the strain energy is assumed to be stored in a few of its members, for example, the columns of a one-story frame building with very stiff girders. For this case, the dynamic load factors give the ratio of internal forces under dynamic and static loads at any point of those members in which the strain energy of deformation is stored.

In the application of the dynamic load factor curves of figures 5.18, 5.19, and 5.21, the value of C_T is determined from the mass, m , the spring constant, k , and the load duration, T , of the equivalent dynamic system. Using C_T , the D.L.F. and t_m/T are determined from figures 5.18, 5.19, and 5.21 for rectangular load, triangular load, and step load with rise time, T_r respectively.

The loading curves determined from the data in EM 1110-345-413 generally do not conform to any of the simplified load shapes for which figures 5.18, 5.19, and 5.21 have been derived. However, with reasonable accuracy, these figures may be used for analysis and design where the given load curves have been approximated by one of these simplified load curves. The method of performing this approximation is explained in paragraph 5-13.

c. Completely Plastic System. For a completely plastic system subjected to the simplified loads shown in figure 5.6, the problem involves five dimensional variables: x_m, R_m, m, B, and T . Only two nondimensional parameters are needed to express the relationship of these variables in chart form, and these parameters are:

$$C_W = \frac{W_m}{W_p} = \text{work done ratio}$$

$$C_R = \frac{R_m}{B} = \text{resistance-load ratio}$$

where

W_m = the maximum work done by the external load and
W_p = the fictitious maximum work done defined in equation (5.12)

The work done ratios for the rectangular and triangular loads, evaluated by the method described in paragraph 5-07c, are plotted in figures 5.22 and 5.23 respectively. The time ratio, t_m/T is also plotted in these figures for each type of loading. These charts indicate that the work done ratio is independent of the duration of the load, but depends on the detail shape of the load; that is, whether the load is rectangular or triangular.

In the application of figures 5.22 and 5.23, the value of W_p and C_R are first determined from the peak load, B , the load duration, T , the mass, m , and the plastic resistance, R_m . Using C_R, C_W is determined from these figures. The maximum displacement, x_m , is equal to W_m/R_m where W_m is the product of C_W and W_p

d. Elasto-Plastic System. There are six dimensional variables, x_m, m, k, R_m, B, and T, involved in the problem of an elasto-plastic system subjected to a rectangular or a triangular load. Hence, three non-dimensional parameters are needed and these are chosen to be

$$C_W = \frac{W_m}{W_p} \qquad C_T = \frac{T}{T_n} \qquad C_R = \frac{R_m}{B}$$

For given values of C_T and C_R, applying the method described in paragraph 5-07d, the displacement function for a given load can be obtained. Substituting the maximum displacement into equation (5.38), the maximum work done by the external load can be evaluated. This procedure was applied to an elasto-plastic system subjected to rectangular and triangular loads. The results thus obtained are plotted in figures 5.24 and 5.27. Because of the importance of the elasto-plastic system in actual application, four more graphs are plotted in figures 5.25, 5.26, 5.28, and 5.29. The first two are for rectangular loads and the last two are for triangular loads.

In figures 5.25 and 5.28, the ordinates are x_m/x_e where x_e is the maximum elastic deflection. These two graphs are useful in determining the maximum displacement of a given structure for a given load.

In both figures 5.26 and 5.29, the ordinate is t_m/T where t_m is the time when the maximum displacement is reached.

In the application of these graphs to an elasto-plastic system, the non-dimensional parameters, C_R and C_T are determined from the variables of the system. Using C_R and C_T, the values of C_W, x_m/x_e and t_m/T are determined from figures 5.24, 5.25, and 5.26 respectively for rectangular load, for figures 5.27, 5.28, and 5.29 for triangular load. The maximum work done, W_m, the maximum displacement, x_m, and the time, t_m, when x_m occurs can then be determined.

Although these graphs are derived for rectangular or triangular loads, they are generally applicable to loads which do not conform to these simplified shapes. For this condition, it is necessary to idealize the given load and this is discussed in paragraph 5-13.

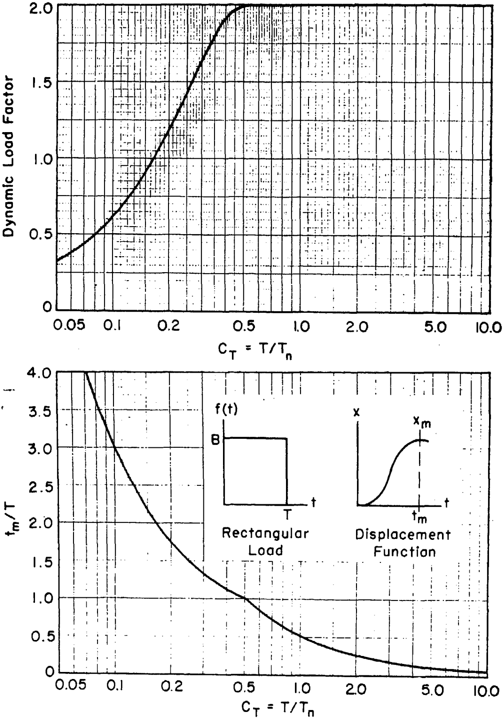
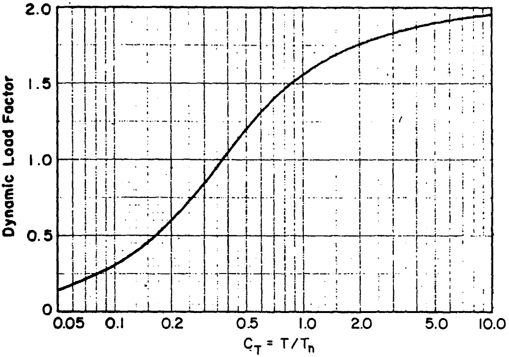


Figure 5.18. Dynamic load factor and t_m/T curves for linearly elastic system, rectangular load



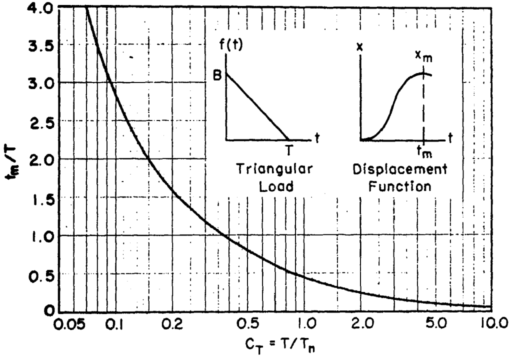


Figure 5.19. Dynamic load factor and t_m/T curves for linearly elastic system, triangular load

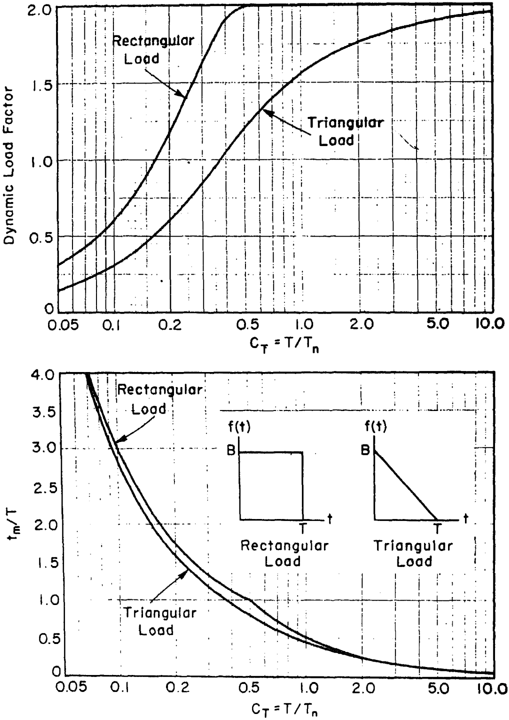


Figure 5.20. Composite dynamic load factor and t_m/T curves for linearly elastic system, rectangular and triangular load

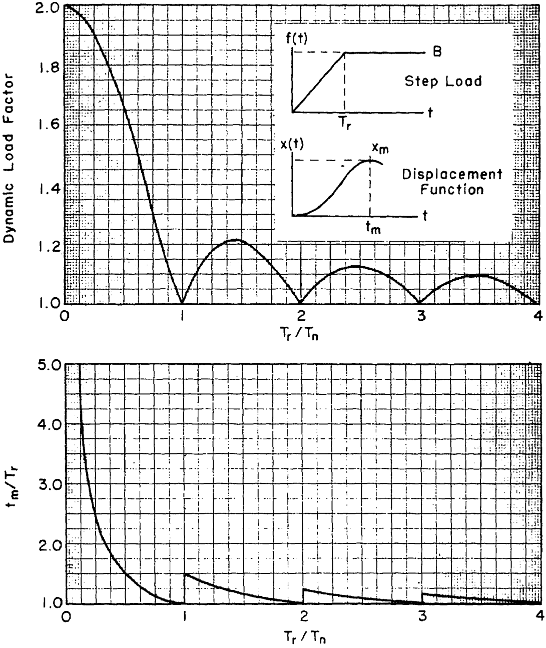


Figure 5.21. Dynamic load factor and t_m/T curves for linearly elastic system, step load with finite rise time, T_r

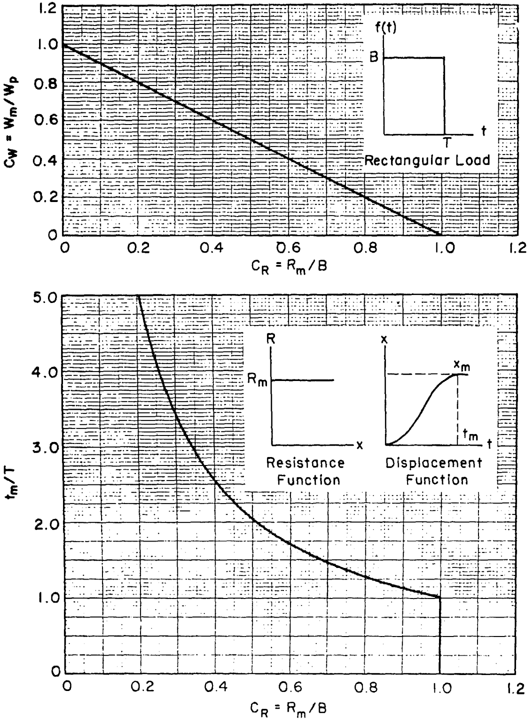


Figure 5.22. Work done ratio and t_m/T curves for completely plastic system, rectangular load

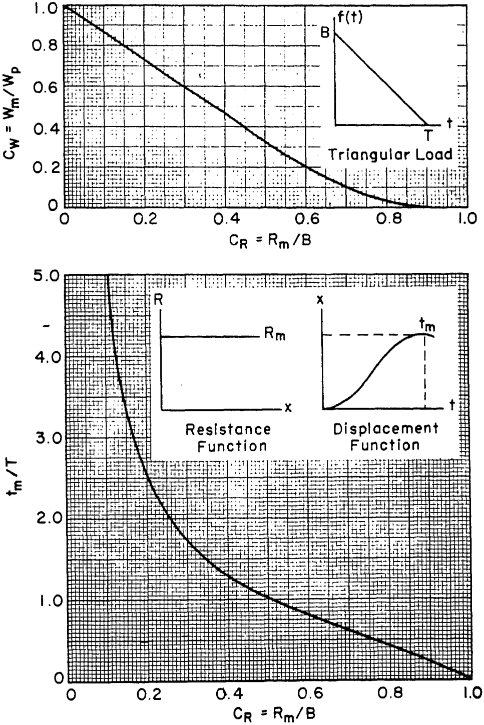


Figure 5.23. Work done ratio and t_m/T curves for completely plastic system, triangular load

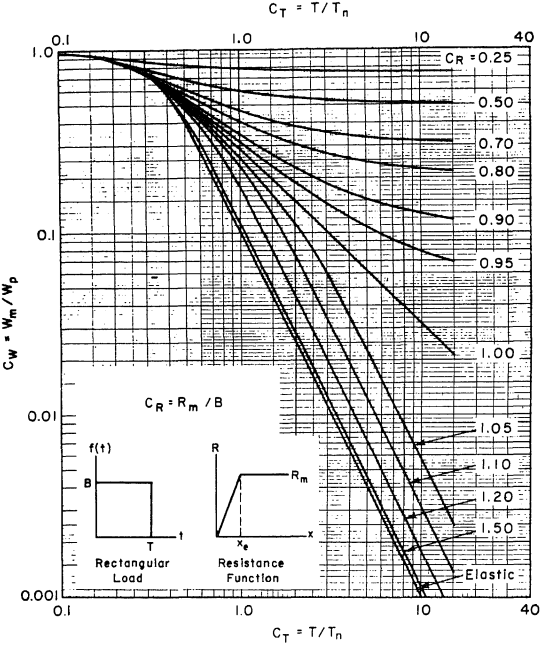


Figure 5.24. Work done ratio for elasto-plastic system, rectangular load

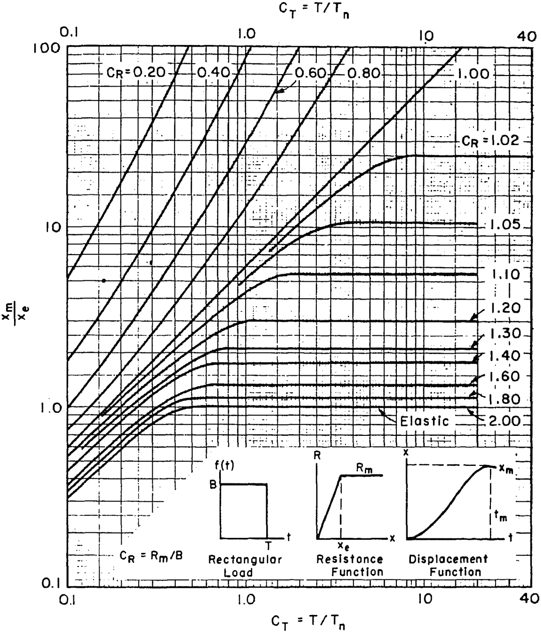


Figure 5.25. $\frac{x_m}{x_e}$ *curves for elasto-plastic system, rectangular load*

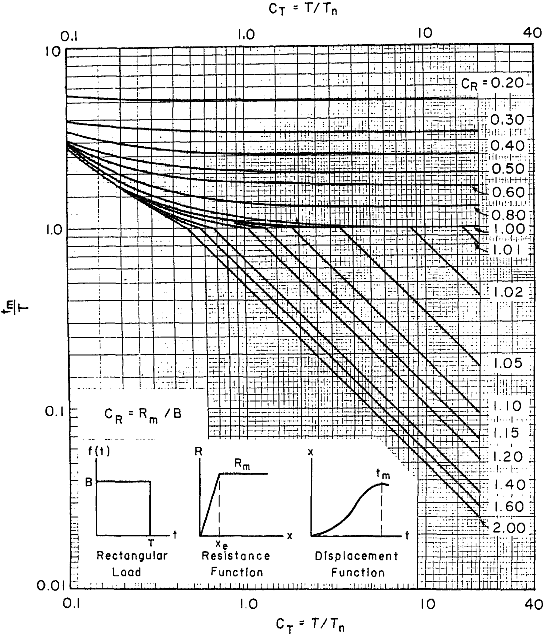


Figure 5.26. $\frac{t_m}{T}$ *curves for elasto-plastic system, rectangular load*

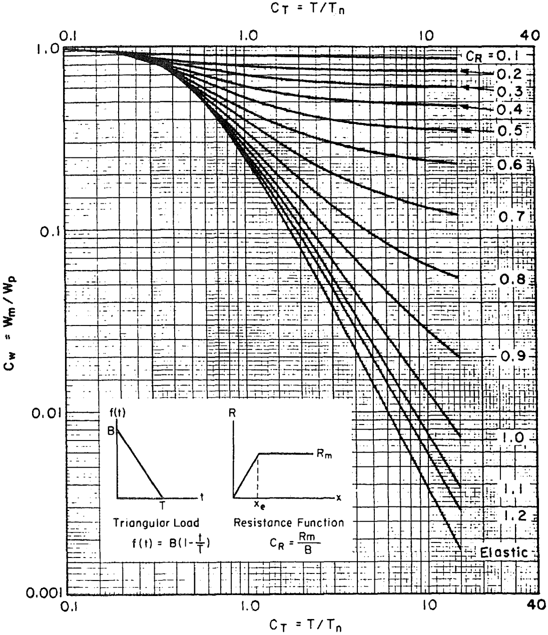


Figure 5.27. Work done ratio for elasto-plastic system, triangular load

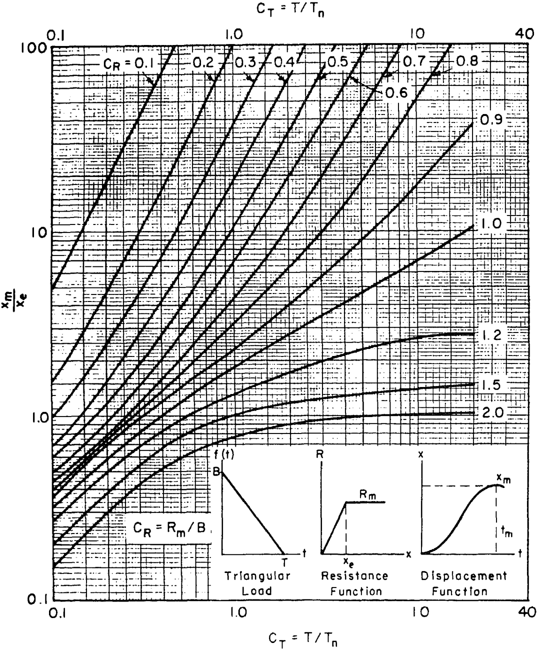


Figure 5.28. $\frac{x_m}{x_e}$ curves for elasto-plastic system, triangular load

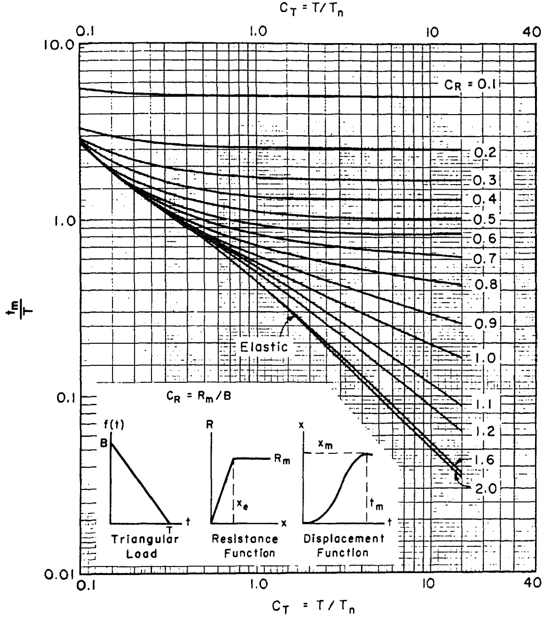


Figure 5.29. $\frac{t_m}{T}$ curves for elasto-plastic system, triangular load

e. Discussion of the Design Charts. In the preceding paragraphs (b, c and d) the response of single-degree dynamic systems subjected to simplified loads are presented in non-dimensional graphs. In actual application, it would be helpful to know how the response of dynamic systems varies with the peak value, shape and duration of the dynamic load and how the response is affected by the mass, spring constant and plastic resistance of the dynamic system. In appendix A, following text, a quantitative discussion of the effects of small increments of the variables, namely: T, m, k, B, and R_m , on the response are presented in detail and these effects are expressed in terms of percentage increment ratios. In this paragraph, only a summary of the results derived in appendix A is described.

First, the response of a linearly elastic system is considered. The dynamic load factor curves for both rectangular and triangular loads are shown in figure 5.20. Depending on the value of C_T , these two curves are separated into three regions. In the first region, where $C_T > 3.0$, the D.L.F. for the rectangular load is constant; and the D.L.F. for the triangular load is practically constant. At a given C_T , the D.L.F. for both types of load are approximately equal to each other. Hence, in this region the maximum displacement, x_m , is: (1) directly proportional to the peak load, B; (2) inversely proportional to the spring constant, k , and (3) practically independent of the mass, m , the load duration, T , and the detail shape of the initially-peaked load. In the second region where $C_T < 0.3$, the D.L.F. for the rectangular load at a given C_T is about twice the corresponding value for a triangular load. This indicates that the area under the load-time curve (the load impulse defined by equation (5.5)) is important while the shape of the load is not important. In this region, the maximum displacement, x_m , is: (1) directly proportional to the peak load, B; (2) directly proportional to the load duration, T; (3) inversely proportional to the square root of the mass, m; and (4) inversely proportional to the square root of the spring constant, k . In the third region, where $0.3 < C_T < 3.0$, all the variables m, B, T, k, and the shape of the load are important. It can therefore be concluded that, so far as the maximum displacement of a linearly elastic system subjected to rectangular or triangular load is concerned, the peak load, B , is always an important variable. The spring constant, k, is also important especially when C_T is large. The mass, m, and the load duration, T, are important only when C_T is small. The detail shape of the load is important when C_T is between 0.3 and 3.0.

Second, the response of a completely plastic system subjected to a rectangular or triangular load is considered. The maximum work done, W_m, is: (1) inversely proportional to the mass, m; (2) proportional to the square of the load duration, T; (3) approximately proportional to the square of the peak load, B; and (4) inversely proportional to the plastic resistance, R_m. The above conclusions are also true for the maximum displacement, x_m, with the exception that the maximum displacement, x_m, is inversely proportional to the square of the plastic resistance, R_m. Therefore it can be concluded that all the variables B, T, m and R_m are always important in the design of a completely plastic system.

Third, the response of an elasto-plastic system subjected to a triangular load as shown in figures 5.27 and 5.28 is considered. Referring to figure 5.27, the work done ratio, C_W, is practically constant when C_T is smaller than 0.2. In this region, the maximum work done is: proportional to the square of the peak load; proportional to the square of the load duration; and inversely proportional to the mass. Hence, the peak load, the load duration and the mass are important variables for short duration load. For a given value of C_R, the work done ratio is also practically a constant for large values of C_T when C_R is small. Thus the peak load, the load duration and the mass are also important variables for long duration loads provided C_R is small. On the other hand for large values of C_T and C_R, the values of C_W decrease with the increase of C_T, hence the spring constant is important under these conditions. The value of C_W also decreases with the increase of C_R, hence the plastic resistance, R_m, is always important especially at large values of C_T.

Similarly, referring to figure 5.28, it can be deduced that, in the determination of the maximum displacement, the peak load is by far the most important variable. The plastic resistance is important for large values of C_T provided $0.5 < C_R < 1.25$. The mass and the load duration are important for short duration loads and those two variables are also important for long duration loads provided C_R is small. The spring constant is important for large values of C_T provided C_R is also large.

DESIGN METHODS FOR SINGLE-DEGREE-OF-FREEDOM DYNAMIC SYSTEMS

5-11 INTRODUCTION. The following paragraphs deal with the design of structures or their members which can be represented by a single-degree dynamic system.

For a given design problem, the dynamic load is always given. Of the three variables of the dynamic system, namely, m, k, R_m, the mass can be estimated and hence is considered to be known. There are also the independent variable, time, and the dependent variable, displacement of the dynamic system, the magnitude of which may not exceed specified limits. Hence, the designer's problem is to design a single-degree dynamic system with a known mass subjected to a given load such that the maximum deflection is equal to or smaller than a certain amount.

There are two variables, k and R_m, of the dynamic system which must be determined to complete the design. Theoretically, there should be an infinite number of combinations of k and R_m to satisfy the given design specifications. However, considerations of economy, type of construction, and standard structural practice limit the relative variation of the spring constant, k, and the ultimate resistance, R_m. Thus, once the type of construction has been established, the size or composition of the structural members may generally be uniquely determined.

In paragraph 5-10, non-dimensional design charts are presented for idealized loads. The methods of design, using these charts, are first discussed in paragraph 5-12. When the given load is different from the idealized loadings, the method of approximating the given load so that the design charts can be used is discussed in paragraph 5-13.

5-12 DESIGN METHODS USING CHARTS - IDEALIZED LOADINGS. Three methods using the design charts are given in this paragraph for the design of single-degree dynamic systems. These are the energy method and the deflection method for cases where plastic deformation is allowed, and the method based upon the dynamic load factor for cases where only elastic deformation is permitted.

a. Design for Plastic Deformation - Energy Method. The energy method for the design of elasto-plastic systems is based upon the fact that the work done on the system up to any time is equal to the sum of the kinetic and strain energies. At the time of maximum displacement, the kinetic energy is zero and the work done by the external load is equal to the strain energies.

When the dynamic load and the mass of the system are given, the fictitious maximum work done on the system by the external load is given by equation (5.55).

$$W_p = \frac{1}{2m}\left[\int_o^T f(t)dt\right]^2 = \frac{H^2}{2m} \tag{5.55}$$

where

H = area under the load-time curve

= BT for rectangular load, or BT/2 for triangular load.

Suppose, using the method which is described later in this section, a trial size of standard structural section is chosen; the value of k and R_m can be determined from the geometrical dimensions of the structure and the data in EM 1110-345-414; then the following non-dimensional parameters are computed:

$$C_T = \frac{T}{T_n} = \frac{T}{2\pi}\sqrt{\frac{k}{m}} \tag{5.56a}$$

$$C_R = \frac{R_m}{B} \tag{5.56b}$$

For the values of T/T_n and C_R, the work done ratio, $C_W = W_m/W_p$ is then obtained from the design charts. Figure 5.24 is used for rectangular load, and figure 5.27 is used for triangular load. The actual maximum work done on the system, W_m, is then computed from the relation $W_m = C_W W_p$.

The maximum allowable strain energy to be stored in the system is equal to the area under the resistance-displacement curve up to the maximum displacement, x_m. For an elasto-plastic system as shown in figure 5.7c if the maximum allowable deflection is x_m , the maximum allowable strain energy storage in the system is

$$SE = R_m \left(x_m - x_e/2\right) \tag{5.57}$$

Comparison of the maximum work done and the maximum allowable energy storage determines the suitability of the selected size.

Based upon the above discussion, a trial and error design procedure is set up for an elasto-plastic system as follows:

Step 1. From the parameters B and T of the given load and the estimated mass, the fictitious maximum work done, W_p , is computed from equation (5.55).

Step 2. A value of W_m/W_p is first assumed and an approximate value for W_m calculated using W_p from step 1. A first approximation of the required maximum plastic resistance, R_m , is determined from equation (5.57) using this approximate value for W_m and neglecting x_e as follows:

$$R_m = W_m/x_m \tag{5.58a}$$

or

$$R_m = \frac{W_m + E}{2\left(x_m - \frac{x_e}{2}\right)} \tag{5.58b}$$

where

x_m is maximum allowable deflection.

Step 3. A trial section is then chosen having a value of R_m approximately equal to the value given by equation (5.58).

Step 4. The actual value of R_m , the spring constant, k , and the maximum elastic deflection, x_e , are then computed from the selected section and the dimensions of the member.

Step 5. Factors C_T and C_R as expressed in equations (5.56a) and (5.56b) are computed.

Step 6. Using the values of C_T and C_R , C_W is determined from the given design chart. Figure 5.24 is used for rectangular load and figure 5.27 is used for triangular load.

Step 7. The actual maximum external work done, W_m , is obtained as the product of W_p (calculated from equation (5.54)) and C_W obtained from the chart.

Step 8. The maximum allowable strain energy storage in the selected trial section is computed by using equation (5.57).

Step 9. The maximum allowable strain energy storage in the section is compared with the actual maximum work done to determine the suitability of the selected section. If the allowable strain energy and maximum work done are not the same, the above procedure from step 2 to 9 should be repeated using equation (5.58b) until a section is obtained which gives the closest agreement between these two quantities. In the successive trial designs, a new value for C_W approximately equal to the value obtained in step 6 of the previous trial design is assumed. If the maximum allowable strain energy is greater than the maximum work done, the member is overdesigned, and a smaller structural section should be tried and vice versa. The final mass of the system should check the estimated mass in step 1.

The above procedure has been set up for an elasto-plastic dynamic system. The same procedure can also be used for the design of linearly elastic systems and completely plastic systems. For a completely plastic system, T_n is equal to zero, hence T/T_n is always infinite. Thus the asymptotic values of C_W for large values of T/T_n should be used. The asymptotic values of C_W for figures 5.24 and 5.27 are plotted in figures 5.22 and 5.23.

For the design of elastic systems, the curve marked "Elastic" in figures 5.24 and 5.27 should be used. In step 2, the approximate value of R_m is computed as $2W_m/x_e$. The value of C_R need not be computed. An elastic design is based on dynamic stresses up to the elastic limit of the material. If it is desired to limit the stress to less than the elastic limit, the computation in step 4 of R_m and x_e should be based on the maximum stress intensity desired in the elastic design. The dynamic load factor method presented in paragraph 5-12c is recommended for elastic design. This method is based on the computation of equivalent static loads or deflections as opposed to energy considerations.

The above procedure gives an exact solution where the external load is either rectangular or triangular, because the charts from which C_W are obtained are computed based on these two types of loads. For any other type of loading which can be completely specified by two parameters,

B and T , similar charts can be prepared. The above procedure would still give the exact solution provided the value of C_W were obtained from the design chart corresponding to the given type of loading. However, in actual problems, the loading is rather complicated and generally cannot be specified by two parameters. It is impractical to prepare design charts which will give the exact solution for any type of loading. Hence, the design charts are only prepared for two basic types of loading. For an actual problem, the loading must be approximated by either a rectangular or a triangular shape. The method of approximating the given load so that the design chart for rectangular or triangular load can be used is discussed in paragraph 5-13.

The application of the energy method of design to actual problems is given in EM 1110-345-416 and 417. In paragraph 6-11, EM 1110-345-416, a numerical example is given to illustrate the design of beams.

b. Design for Plastic Deformation - Deflection Method. The deflection method of design is practically the same as the energy method with the exception that the non-dimensional parameter, x_m/x_e , is used instead of the work done ratio, C_W . The design procedure for the deflection method is as follows:

Step 1. A value of $C_R = R_m/B$ is assumed, and a first approximate value of the resistance, R_m , determined.

$$R_m = C_R B \tag{5.59}$$

Step 2. A trial section is chosen having a plastic resistance, R_m , approximately equal to the value given by equation (5.59).

Step 3. The actual values of R_m , the spring constant, k , and the maximum elastic deflection, x_e , are computed from the selected section and the dimensions of the member. The mass, m , of the system is also determined.

Step 4. Factors C_T and C_R as expressed in equations (5.56a) and (5.56b) are computed.

Step 5. x_m/x_e is determined from the given design charts. Figure 5.25 is used for rectangular loads, and figure 5.28 is used for triangular loads.

Step 6. The maximum deflection, x_m , is obtained as the product of x_e and x_m/x_e .

Step 7. Comparison of the computed maximum deflection, x_m , with the design specification for the maximum allowable x_m determines the suitability of the selected trial section. By assuming a new value of C_R , the design procedure from steps 1 to 7 is repeated until a section which gives the closest agreement between the computed x_m and the allowable x_m is obtained.

c. Design for Elastic Deformation - Dynamic Load Factor Method. This design procedure is for the case in which the external load can be approximated by either a rectangular or triangular shape or a step load with a finite rise time as covered by the design charts in figures 5.18, 5.19, and 5.21. Methods for approximating a given load to conform to one of these simple types are covered in paragraph 5-13.

Step 1. A value of dynamic load factor is first assumed.

Step 2. A trial section is selected having a maximum resistance, R_m , approximately equal to

$$R_m \; (\text{D.L.F.})B$$

where

D.L.F. is the assumed dynamic load factor.

Step 3. The spring constant, k , is determined as described in paragraph b above, and the maximum elastic deflection is determined for various type elements as described in paragraphs 6-13 through 6-24 in EM 1110-345-416.

Step 4. The time ratio, $C_T = T/T_n = \frac{T}{2\pi}\sqrt{\frac{k}{m}}$ is determined.

Step 5. Using C_T , a revised value for the dynamic load factor is determined from figures 5.18 or 5.19 for rectangular or triangular loads respectively, or from figure 5.21 for step loads with a finite rise time.

Step 6. The static deflection, x_s , is determined by $x_s = B/k$.

Step 7. The maximum deflection, x_m , is obtained as the product of x_s and the revised dynamic load factor.

Step 8. Comparison of x_m and x_e determines the suitability of the selected trial section. The design procedure from steps 1 to 8 is repeated until a section which gives the closest agreement between x_m

and x_e is obtained. If it is desired to limit the maximum deflection to less than the maximum elastic deflection, a fictitious x_e equal to the desired maximum deflection can be used. The maximum resistance of the section, R_m , would be reduced in the same ratio as x_e to correspond with the smaller design deflection.

An alternate dynamic load factor method may be used if desired, in which stresses are used rather than deflections. To use this approach, proceed as above but omit computation of x_e, x_s, and x_m . For step 8 compare R_m of section with (D.L.F.)B to determine suitability of the selected trial section. By either approach, it is necessary to determine t_m and check the load shape as discussed in paragraph 5-13.

The application of this method to the design of elastic beams and one-story buildings is illustrated by numerical examples in EM 1110-345-417.

5-13 METHOD OF APPROXIMATING A GIVEN LOAD. The design methods given in the previous paragraph are based upon an idealized load acting on a dynamic system with an idealized resistance function. For an actual problem, the resistance function is very close to one of the idealized forms for an elastic or an elasto-plastic system; however, the given load may be radically different from either a rectangular load or a triangular load or a step load with a finite rise time. Hence, the first step in the design is the replacement of the given load by one of the idealized loads for which the design charts are provided.

The basic consideration in replacing the given loading by an idealized loading shape is that the maximum displacement produced by both must be equal. This replacement of load usually introduces a certain amount of error and is one of the more difficult parts of the whole design problem. The amount of error depends largely upon how well the loading is approximated which depends entirely upon the judgement of the designer. No general rules can be given for replacing a given load by a simplified load; however, the following is set up as a guide.

Suppose for a given problem, the load is shown in figure 5.30a. In the first approximation a rectangular load as shown by the dotted curve may be used. Following the design procedures set up in paragraph 5-12, the values of C_T and C_R are obtained. Then the value of t_m/T is obtained from figure 5.18, 5.22, or 5.26. This parameter is very important in judging how good the load approximation is. Since t_m is the time at which x_m occurs, the approximate load and the given load should (1) be very close to each other in the time interval of time from 0 to t_m, and (2) have about the same area in this time interval.

Depending upon the value of t_m obtained from the first approximation, the given load should be replaced by one of the rectangular loads shown in figure 5.30b and figure 5.30c. However, if t_m is much larger than T_2 as shown in figure 5.30a, it is expected that the internal resisting force in the time interval from 0 to T_2 is small. Based on the discussion in paragraph 5-05a, for this case, the detail shape of the load is not important, and the given load can be replaced by a pure pulse load with equal total impulse. In this case, the value of C_W is approximately equal to unity, hence $W_m = W_p$

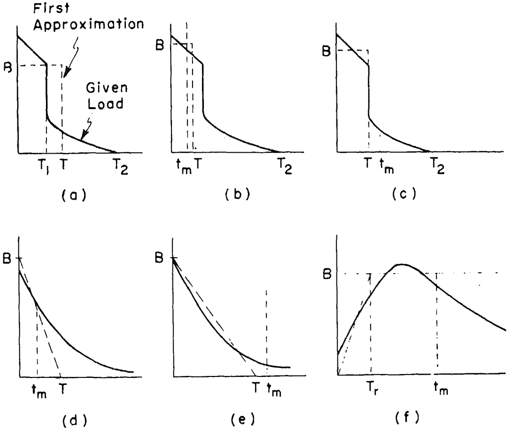


Figure 5.30. Typical load approximations

In figure 5.30a, b, c, a rectangular load is used to replace the given load, hence the value of t_m is obtained from figures 5.18, 5.22, or 5.26. For the case when the given load is replaced by a triangular load (figures 5.30d and 5.30e), t_m may be obtained from figures 5.19, 5.23, or 5.29. When the given load is replaced by a load with a finite rise time (figure 5.30f) t_m may be obtained from figure 5.21.

5-14 DISCUSSION OF DESIGN METHODS. a. Accuracy of Design Methods. From the above discussion, it is seen that in applying the design procedures given in paragraph 5-12 to an actual problem, idealized load and resistance values have to be used to replace the actual values. It is expected that a certain amount of error will be involved. A final check of the actual design using the numerical methods of analysis described in paragraph 5-08 is necessary. Therefore, this design method can be considered as a preliminary design procedure. The result obtained from the preliminary design must be checked by a numerical method of analysis using the actual load and resistance function.

The maximum displacement evaluated by the numerical method may be different from the design specification. There are two reasons for this discrepancy. One is the fact that idealized loads and resistances are used in the preliminary design. The error from this source is usually less than 5%. The second is due to the use of standard structural sections. Only a limited number of sections are available. The difference in properties between two adjacent sections may be more than 10%. For a given design problem, it is sometimes found that no standard section is available in order to satisfy the given design specifications. If a standard section close to the desired section is used, the maximum displacement may be different from the design specifications by more than 10%. However, a number of design problems have been carried out and it has been found that the accuracy of the preliminary design method, judging by the difference between the actual and the specified maximum displacements, is within 10% for the design of one-story buildings, and 25% for the design of structural members, such as beams or slabs. The larger error in the design of beams and slabs is due to their shorter periods which make them more sensitive to the shape of the load, and to gaps in available structural sections (see appendix A following the text of this manual).

When designing systems with an elasto-plastic behavior, a small change in the maximum resistance of the section will cause proportionately larger changes in the maximum displacement (see appendix A). Because of this sensitivity, a 10% error in the displacement represents a relatively smaller change in the required maximum resistance. Therefore, numerical analysis is not always justified in checking the design of an elasto-plastic system.

b. Comparison of Different Methods of Design. In paragraph 5-12, three different methods of design are given. The energy and deflection methods are applicable to the design for plastic deformation while the dynamic load factor method is applicable only to the design of elastic systems.

In both the deflection and the dynamic load factor methods, the non-dimensional parameter used is a ratio of deflections, x_m/x_e, for the former, and x_m/x_s for the latter. Hence, the dynamic load factor method is actually only a special case of the deflection method.

Basically, there are two different methods of design, the energy method, and the deflection method. The design procedures for these two methods as given in paragraph 5-12 are on a trial and error basis. After a trial section is selected, the amount of numerical work involved in determining the suitability of the selected section is about the same no matter which method is used. But the convergence of successive trials for a given problem depends on which method is used.

In the design of one-story buildings which can be replaced by a single-degree dynamic system, the value of C_T is of the order of unity; and the value of C_R varies from 0.25 to 1.5 for an elasto-plastic system under triangular load. In these ranges of values for C_R and C_T, the maximum work done, W_m, is not very sensitive to small variations in the spring constant, k, or the plastic resistance, R_m (appendix A). In other words, the maximum work done is practically constant when different sections are used in a trial and error design procedure. Thus, in successive trials, the only modification is in the strain energy storage. If in one trial design the section is found to be unsatisfactory, a second trial can be started by using another section with maximum strain energy storage approximately equal to the work done obtained in the previous trial design. On the other hand, if the deflection method is used within the same ranges of values of C_R and T/T_n as mentioned before, the maximum displacement is rather sensitive to any change in the value of the spring constant, k , or plastic resistance, R_m . If a trial design is found to be unsatisfactory, there is practically no guide as to what the next trial section should be. Hence, in the design of one-story buildings, the energy method of preliminary design is preferred to the deflection method.

However, in the design of individual members such as beams or slabs under triangular load, the value of C_T is usually greater than 3 and the value of C_R is usually greater than 1. In this range of values of C_T and C_R , the variation of W_m with any change in k or R_m is about of the same order of magnitude as the variation of x_m (appendix A of this manual). Hence, in the design of beams or slabs, the deflection method is as convenient as the energy method so far as the convergence of successive trials is concerned.

So far the discussion has been limited to the design of a single-degree dynamic system. For a multi-degree dynamic system such as a multi-story building, the design problem is complicated by the inter-actions between different stories. It is shown in paragraph 5-21c that, for design purposes, any story of a multi-story building can be replaced by an equivalent single-degree dynamic system. From this, a method of design based on energy considerations has been developed which gives reasonably good results.

ANALYSIS OF MULTI-DEGREE-OF-FREEDOM DYNAMIC SYSTEMS

5-15 INTRODUCTION. In the following paragraphs, the general problem of analysis of multi-degree-of-freedom dynamic systems is considered. The types of structures and structural members which are replaced by multi-degree dynamic systems for analysis are, for example: (1) beams under lateral load when higher modes as well as the fundamental mode are considered (appendix A of EM 1110-345-416); (2) the effect of support movement on the response of a beam; and (3) multi-degree buildings under lateral loads.

In order to make the problems under discussion easier to be visualized, the terminology of multi-story buildings is used in this and the subsequent sections of this manual. It should be understood that the methods of analysis are applicable for any multi-degree dynamic system and are by no means limited to multi-story buildings. In paragraph 5-16, the equations of motion are set up. The rigorous and numerical methods of analysis are given in paragraphs 5-17 and 5-18 respectively.

5-16 EQUATIONS OF MOTION. For the dynamic analysis of a multi-story building of n stories acted upon by lateral blast loads, it is necessary to replace the structure by an equivalent dynamic system. The distributed masses of the structure are lumped together at the floor and roof levels, giving a series of n lumped masses. The masses are connected to each other by weightless springs which simulate the lateral resistance of the structure. Similarly the distributed loads on the structure are lumped together at the lumped masses. Hence, the multi-story building of n stories is represented by an n-th degree dynamic system with n concentrated masses connected by weightless springs and acted upon by concentrated loads at each of the masses. The variables under consideration are the displacements of these masses.

The following notations are used:

For masses: $m_1, m_2 \ldots . m_{g-1}, m_g, m_{g+1} \ldots . m_n$

For absolute displacement of these masses:

$$x_1, x_2 \ldots . x_{g-1}, x_g, x_{g+1} \ldots . x_n$$

For external load on these masses:

$$f_1(t), \quad f_2(t) \ldots . f_{g-1}(t), \ f_g(t), \quad f_{g+1}(t) \ldots . f_n(t)$$

The equation of motion of mass, m_g, is given by equation (5.60):

$$m_g \frac{d^2x_g}{dt^2} = f_g(t) - R_{gt} \tag{5.60}$$

where R_{gt} is the total resisting force which is exerted by the columns and walls above and below upon the mass m_g. Depending upon the type of construction, two cases are considered for the determination of R_{gt}. For example, in the case of a shear wall structure, where the building is considered as a vertical cantilever and the effects of over-all bending on the building is taken into consideration, the resistance function, R_{gt}, depends upon the displacements of all the n masses. In the case of frame structures where the effect of over-all bending of the building is neglected, R_{gt} is a function of only three absolute displacements, x_{g-1}, x_g, and x_{g+1}.

a. Multi-Story Shear Wall Buildings. The resisting forces acting on any floor of a multi-story shear wall building depend on the displacements of all the floors. Figure 5.31 shows the absolute displacements of any two floors represented by two masses, m_g and m_i of a multi-story building. The absolute displacement of each mass is represented by x_g and x_i respectively, their relative displacement being given as $x_g - x_i$. Due to the difference in magnitudes of the displacements x_i and x_g, a pair of internal forces, R_{gi}, as shown in figure 5.31 are developed. The masses m_i and m_g are said to be coupled together. If the coupling spring constant is k_{gi}, the expression for the resistance function, R_{gi}, in the elastic range is given by equation (5.61).

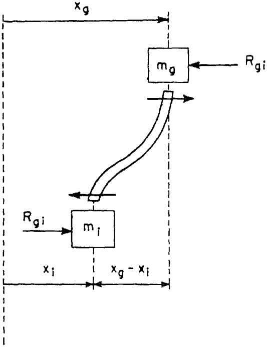


Figure 5.31. Any two floors of a multi-story building

$$R_{gi} = k_{gi}\left(x_g - x_i\right) \qquad (5.61)$$

The positive direction for R_{gi} as well as the positive directions for x_i and x_g are indicated by arrows in figure 5.31.

For a shear wall building, the coupling of each floor mass to every other floor mass and the ground must be considered. The general expression for the total resisting force, R_{gt}, acting on the g-th floor mass of a multi-story shear wall building is:

$$R_{gt} = R_{go} + R_{g1} + \dots + R_{gi} + \dots + R_{gn} = \sum_{i=0}^{n} R_{gi} \qquad (5.62)$$

The design of multi-story shear wall buildings will generally be simplified to elastic analysis because of the small allowable lateral deflections and the complexity of the problem. For this case, the above expression may be written as:

$$R_{gt} = k_{go}x_g + k_{g1}\left(x_g - x_1\right) + \dots k_{gi}\left(x_g - x_i\right) \dots + k_{gn}\left(x_g - x_n\right)$$

$$\sum_{i=0}^{n} k_{gi}\left(x_g - x_i\right) \qquad (5.63)*$$

In equation (5.62), R_{go} is the resistance developed by the shear wall at the g-th floor due to the relative movement of m_g with respect to the ground. Note that when $i = g$, $R_{gi} = R_{gg} = 0$. In equation (5.63), the ground displacement is assumed to be zero.

Substituting equation (5.62) into equation (5.60), the equation of motion for mass, m_g, is:

$$m_g \frac{d^2x_g}{dt^2} = f_g(t) - \sum_{i=0}^{n} R_{gi} \qquad (5.64)$$

In the elastic range, equation (5.64) becomes

$$m_g \frac{d^2x_g}{dt^2} = f_g(t) - \sum_{i=0}^{n} k_{gi}\left(x_g - x_i\right) \qquad (5.65)$$

There is one equation of the same type as equation (5.64) for each floor mass (see figure 5.32). Hence, the equations of motion for an n-story building consist of n simultaneous second order differential equations.

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* The resistance function, R_{gt}, may also be expressed by:

$$R_{gt} = k'_{g1}x_1 + k'_{g2}x_2 + \dots k'_{gn}x_n = \sum_{i=1}^{n} k'_{gi}x_i; \text{ where } k'_{gi} = -k_{gi} \text{ and}$$

$$k'_{gg} = k_{go} + k_{g1} + k_{g2} + k_{g3} \dots k_{gi} + \dots k_{gn}.$$

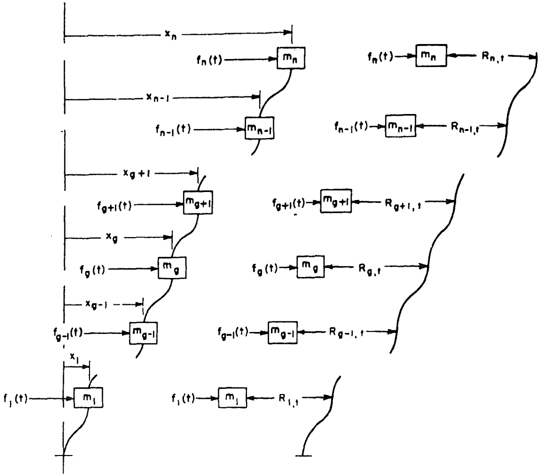


Figure 5.32. Equivalent dynamic system for an n-story shear wall building

For a multi-story shear wall building in the elastic range, the variables of the equivalent dynamic system can be evaluated based on methods which are given in detail in EM 1110-345-416 and EM 1110-345-419.

Having determined the variables of the system, and the load acting on the structure, the analysis is carried out for the elastic range using equations of the form 5.65, as explained in paragraphs 5-17 and 5-18.

As soon as the stress at any point of a shear wall for any story exceeds the elastic limit, not only the internal resisting force in this story is affected, but also practically all of the coupling constants of the equivalent dynamic system are affected. Whenever the stress in any story exceeds the elastic limit, the resistance functions of the complete structure have to be re-computed. Hence, it would not be practical to attempt to analyze a multi-story shear wall building with stresses exceeding the elastic limit. The application of the above equation is therefore limited to the analysis of shear wall buildings in the elastic range.

b. Multi-Story Frame Buildings.

In frame buildings, most of the resistances, R_{gi}, in equation (5.62) are either zero or negligibly small except for the resistances, $R_{g,\ g+1}$, and $R_{g,\ g-1}$, which are the resistances developed between adjacent masses. In an actual building, these resistances are developed between the floors by the columns and walls. Because of the relatively large stiffness of the frame elements in the plane of the floor in frame buildings, the lateral restraints are not so important between non-adjacent floor levels. Eliminating all resistances except those developed between adjacent masses, the equivalent system in figure 5.33 is obtained. This equivalent system, which is applicable for both elastic and elasto-plastic behavior, is used in the analysis of multi-story frame buildings.

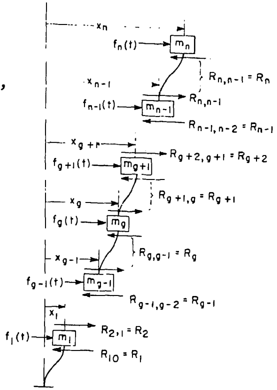


Figure 5.33. Equivalent dynamic system for a multi-story frame building

For the equivalent system shown in figure 5.33, using the following simplified notations:

$$R_{10} = R_1; \qquad R_{g,\ g-1} = R_g; \qquad k_{10} = k_1; \qquad k_{g,\ g-1} = k_g$$

the total resisting force, R_{gt}, on mass, m_g, is

$$R_{gt} = R_g - R_{g+1} \tag{5.66}$$

In the elastic range, equation (5.66) becomes

$$R_{gt} = k_g\left(x_g - x_{g-1}\right) + k_{g+1}\left(x_{g+1} - x_g\right) \tag{5.67}$$

Substituting equation (5.66) into equation (5.60), the equation of motion for mass, m_g, is given by

$$m_g \frac{d^2 x_g}{dt^2} = f_g(t) - R_g + R_{g+1} \tag{5.68}$$

In the elastic range, equation (5.68) becomes

$$m_g \frac{d^2 x_g}{dt^2} = f_g(t) - k_g\left(x_g - x_{g-1}\right) + k_{g+1}\left(x_{g+1} - x_g\right) \tag{5.69}$$

Rearranging terms in equation (5.69), the following recurrence formula expressing the relationship of any three consecutive displacement variables is obtained. This is given by

$$-k_g x_{g-1} + m_g \frac{d^2 x_g}{dt^2} + \left(k_g + k_{g+1}\right)x_g - k_{g+1}x_{g+1} = f_g(t) \tag{5.70}$$

For an n-th degree dynamic system as shown in figure 5.33, there are n separate equations of the same type as equation (5.70). These n equations are

$$\begin{aligned}
&m_1 \frac{d^2 x_1}{dt^2} + \left(k_1 + k_2\right)x_1 - k_2 x_2 = f_1(t) \\
&m_2 \frac{d^2 x_2}{dt^2} + \left(k_2 + k_3\right)x_2 - k_2 x_1 - k_3 x_3 = f_2(t) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&m_g \frac{d^2 x_g}{dt^2} + \left(k_g + k_{g+1}\right)x_g - k_g x_{g-1} - k_{g+1}x_{g+1} = f_g(t) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{5.71}$$

$$m_n \frac{d^2x_n}{dt^2} + k_n x_n - k_n x_{n-1} = f_n(t)$$

In equation (5.71), all the coupling springs are assumed to be in the elastic range. If a certain spring such as k_g has reached its plastic resistance, R_{gm}, this value should be used for R_g instead of $\left[k_g\left(x_g - x_{g-1}\right)\right]$.

The analysis of an n-th degree dynamic system involves the evaluation the n simultaneous differential equations.

5-17 RIGOROUS METHOD OF ANALYSIS. In the rigorous method, the standard procedures of solving simultaneous differential equations are used. Load and resistance functions must be expressed in simple mathematical forms. The displacement of any mass, say x_g, is first evaluated. The remaining x's are then determined using the recurrence formula given by equation (5.70) for frame buildings, or equation (5.65) for shear wall buildings.

Eliminating all the x's except x_g from the simultaneous equations given by equation (5.71) for frame buildings or equation (5.65) for shear wall buildings, a linear differential equation of the (2n)th order in x_g is obtained. Just like the case of single-degree dynamic systems, the solution of x_g of an n-th degree dynamic system consists of two parts. One is the transient term and the other is the forced solution. Thus the complete solution of x_g can be expressed as

$$x_g = \left[\sum_{i=1}^{n} C_{gi} \sin\left(2\pi \frac{t}{T_{ni}} - \phi_{gi}\right)\right] + x_{gf} \tag{5.72}$$

where x_g is the total displacement of the mass, m_g; x_{gf} is the portion of x_g which is referred to as the forced solution; and the term in brackets is the portion of x_g referred to as the transient solution. There are 2n constants of integration in the complete solution. These constants are determined from the initial conditions that all the n masses are at rest at the initiation of the load.

There are n oscillatory terms in the transient solution. The frequencies of oscillation are the n natural frequencies of the n-th degree dynamic system. The lowest frequency is defined as the fundamental frequency of the dynamic system.

Several other methods, based on the same principle as described above, but differing in detailed techniques are also commonly used in the analysis of multi-degree dynamic systems. In one method, the set of simultaneous differential equations (5.71) or (5.65) is first transformed into a set of algebraic equations by applying the principle of Laplace Transformation and operational calculus [4]. The complete solution is then obtained by inverse transformation. In another method, the solution of the problem is started from the recurrence formula, equation (5.70), by applying the methods of solving finite difference equations [8].

There is another method which is radically different from the above methods. In this case, the principle of generalized coordinates is used [2, 6, 7]. An n-th degree dynamic system is changed to an equivalent system with n uncoupled masses. Each mass is individually supported by a spring and acted upon by a generalized load. Thus the n-th degree dynamic system is changed to a system of n uncoupled single-degree dynamic systems. This method of generalized coordinates is extensively used in the study of structural members such as beams, columns, or slabs, when the high modes of oscillation are to be considered. The application of this method to the analysis of elastic beams is given in appendix A, EM 1110-345-416, and to the analysis of multi-story frame buildings in reference 6.

The difficulty in applying the rigorous method of analysis to multi-degree dynamic systems is evident. The problem involves the solution of n simultaneous differential equations of second order with 2n initial conditions. Suppose two separate equations are needed for each load and the dynamic system is elasto-plastic; then, as many as 4n sets of equations are needed to express the equations of motion. In each set, there are n simultaneous equations. Thus, in general, the rigorous method of analysis is impractical for multi-degree dynamic systems, and numerical methods must be used. The two numerical methods described in paragraph 5-08 for single-degree dynamic systems are also applicable to multi-degree systems. These two methods are described separately in paragraph 5-18.

5-18 NUMERICAL METHODS OF ANALYSIS. a. Linear Acceleration Extrapolation Method. The equation of motion for mass, m_g, is given by equation (5.60). Suppose a_g denotes the acceleration of m_g; the expression for a_g as a function of time is

$$a_g = \frac{d^2x_g}{dt^2} = \frac{1}{m_g}\left[f_g(t) - R_{gt}(t)\right] \tag{5.73}$$

where $R_{gt}(t)$ is the total internal resisting force acting upon m_g at any time, t. The expressions of R_{gt} for the shear wall building and frame building discussed in paragraph 5-16 are given respectively by the following equations:

$$\text{Shear wall building,}\quad R_{gt} = \sum_{i=o}^{n} R_{gi} \tag{5.74}$$

$$\text{Frame building,}\quad R_{gt} = R_g - R_{g+1} \tag{5.75}$$

If the acceleration a_g is assumed to be linear in the time interval from t_n to t_{n+1}, from equation (5.44), the relationship between the displacements of x_g at t_n and t_{n+1} is given by

$$\left(x_g\right)_{t=t_{n+1}} = \left(x_g\right)_{t=t_n} + \left(v_g\right)_{t=t_n}(\Delta t) + \left[\frac{1}{3}\left(a_g\right)_{t=t_n} + \frac{1}{6}\left(a_g\right)_{t=t_{n+1}}\right](\Delta t)^2 \tag{5.76}$$

where

$$\left(v_g\right)_{t=t_n} = \left(\frac{dx_g}{dt}\right)_{t=t_n}$$

$$\Delta t = t_{n+1} - t_n$$

Based upon equation (5.76), a trial and error procedure can be set up for the computation of x's at t_{n+1} from the values of x's at t_n. A set of n values of $x_1, x_2, \ldots x_g, \ldots x_n$ are assumed for the displacement of each mass at time t_{n+1} from which the resistance R's and the acceleration a's at time t_{n+1} are determined. The actual values of x's at t_{n+1} are then computed from equation (5.76). Comparison of the computed and assumed values of x's indicates the correctness of the assumed values.

This trial and error procedure is tedious to carry out since it involves n variables. The computed values of all the n variables must agree respectively with the n assumed values. On the other hand, this method usually gives accurate results. When all the coupling springs are in the plastic range where the resistances are constant, the computation becomes straightforward. Further details of this method are given in paragraph 5-08 where it is described for the analysis of a single-degree dynamic system.

b. Acceleration Impulse Extrapolation Method. The recurrence formula, equation (5.49), for the single-degree dynamic system is also applicable for multi-degree dynamic systems. For the displacement, x_g, of mass, m_g, the recurrence formula is

$$\left(x_g\right)_{t = t_{n+1}} = 2\left(x_g\right)_{t = t_n} - \left(x_g\right)_{t = t_{n-1}} + \left(a_g\right)_{t = t_n} (\Delta t)^2 \qquad (5.77)$$

where

$\Delta t = t_{n+1} - t_n = t_n - t_{n-1} = \dots = t_2 - t_1$

$\left(a_g\right)_{t = t_n}$ = the acceleration of mass, m_g, at time, $t = t_n$, the expression given by equation (5.73).

The method of modifying the acceleration at any point of discontinuity of the acceleration curve is the same as in the case of single-degree dynamic systems (see paragraph 5-08). The advantage and disadvantage of this method as described for the single-degree dynamic system also hold for multi-degree systems. If the time interval (Δt) is chosen to be about $T_n/10$ where T_n is the shortest period of oscillation, the accuracy in the maximum absolute displacement is about 3%.

The application of the acceleration impulse extrapolation method to the analysis of multi-story frame buildings is shown in EM 1110-345-418. The equivalent dynamic system for frame buildings is the case shown in figure 5.33.

DESIGN OF MULTI-STORY FRAME BUILDINGS FOR PLASTIC DEFORMATION

5-19 INTRODUCTION. The problem considered in this and the following paragraph is the design of a multi-story frame building which is represented by the equivalent dynamic system shown in figure 5.33. The terminologies of multi-story buildings are used in the discussion. Thus the concentrated masses are approximately the masses of different floors, the displacements of each mass are the displacement of each floor level, and the springs are the columns between adjacent floors.

In the design problem, the floor masses can be estimated and the external loads acting on the masses are given. The maximum allowable displacements between adjacent floors are prescribed. The problem is the selection of the column sections for each story such that the relative displacements between adjacent floors do not exceed the prescribed maximum allowable values.

One approach to the design problem is the preparation of non-dimensional design charts as in the case of single-degree dynamic systems. Once charts are available, the complete design can be carried out in a series of routine computations. A little reflection will indicate that, even for a two-story building, there are too many non-dimensional parameters to permit the preparation of practical design charts to cover all possible cases. Thus the design of multi-story buildings cannot be carried out in as simple and straightforward a manner as the design of single-degree dynamic systems.

Another method of approach to the design problem is to first assume the sizes of all the columns for the entire building and then check the adequacy of the assumed sizes by a numerical analysis of the whole structure. Any readjustment of the column size is then made according to the results of the numerical analysis. The size adjustment is difficult because the change of column size of any one-story affects the relative displacements of the entire structure. If, in the numerical analysis, the relative displacements between several adjacent floors are found to be unsatisfactory, it is difficult to decide which column needs modification, and by what amount the size of the column should be changed. The detailed numerical analysis may have to be carried out over and over again before a set of satisfactory column sizes is found. This method, therefore, is difficult and tedious to carry out, and is not recommended except for problems where a rational method of design has not been developed.

The design method recommended for multi-story frame buildings is based on a floor-by-floor iteration procedure in which the column size for the first, second, third, and on up to the top story is selected one after the other. Specifically, the outline of the design procedure is as follows:

Step 1. The plastic resistance and the size of the columns for the first story are determined on the assumption that (1) the first floor is a single-degree dynamic system with a mass, m_1, subjected to an external load, $f_1(t)$, and (2) all the remaining floors are absent.

Step 2. The plastic resistance, R_{2m}, of the columns for the second story is determined by (1) incorporating the result of the preliminary design of the first story, and (2) assuming that all those floors above the second floor are absent.

Step 3. The plastic resistance, R_{3m}, of the columns for the third story is determined by (1) incorporating the results of the preliminary design of the first and second stories, and (2) assuming that all those floors above the third floor are absent.

Step 4. The above procedure is continued until the ultimate resistance, R_{nm}, of the columns for the top story is determined.

Step 5. The preliminary plastic resistances determined in step 1 through 4 for each story are adjusted by taking into account the effect on the columns of a given story by the columns of the story which are directly above the given story. For example, the shearing force developed in the columns of the third story is considered in modifying the plastic resistance of the second story.

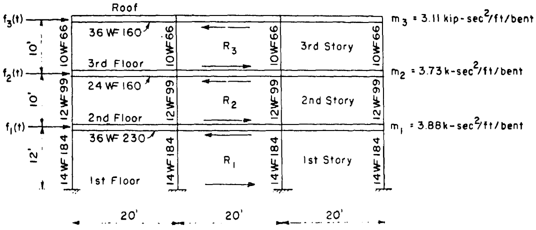
Step 6. The column sizes for different stories are then determined. The sizes of the selected columns should be based on the revised resistance values obtained in step 5. In the determination of the size of the selected columns, the effect of vertical load on the plastic moment of the section should be taken into account.

Step 7. A numerical analysis for the entire building is then carried out to determine the suitability of the selected columns. Any revision of the column size can be made according to the result of the numerical analysis.

There are two basic procedures in the above floor-by-floor iteration design method. One is the preliminary determination of the resistance of the columns for any story after the column resistances of all those stories below this story have been determined. The other is the adjustment of the preliminary resistance thus determined for any story after the column resistances of all those stories above have been determined. The fundamental principles in carrying out these two basic procedures are discussed in paragraphs 5-21 and 5-22 respectively. The complete design procedure for the design of multi-story buildings is given in detail in paragraph 5-23. The assumptions used in the floor-by-floor design procedure and the accuracy of this design method are summarized in paragraph 5-25.

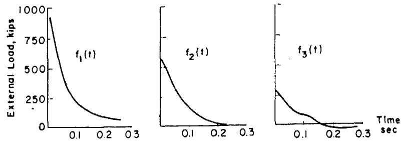
In paragraph 5-20, the absolute displacements, relative displacements and resistance functions of a three-story building are plotted versus time. These curves are used (1) to illustrate the general characteristics of multi-story buildings, and (2) to explain why a somewhat complicated procedure has to be used in design.

5-20 DISPLACEMENTS AND RESISTANCE FUNCTIONS OF A THREE-STORY FRAME BUILDING. A three-story frame building designed for blast resistance is illustrated in figure 5.34. The floor-by-floor resistances and displacements of this building under the design blast loading are plotted in figure 5.35. This building was designed based on the specification that

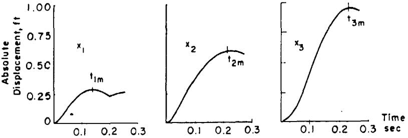


18' c. to c. Bents, Bldg. Length = 180'
Concrete Slab Walls, Roof and Floors

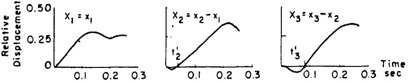
Figure 5.34. Three-story frame building designed for blast resistance



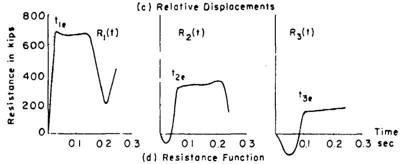
(a) External Loads



(b) Absolute Displacements



(c) Relative Displacements



(d) Resistance Function

Figure 5.35. External loads, displacements and resistances of three-story frame building as functions of time

the maximum allowable relative displacement for any story should not exceed 0.33 ft. Subscripts 1, 2, and 3 refer to the first, second, and third floors above the ground respectively. Hence, the third "floor" is the roof.

Due to the differences in the floor masses and in the external loads on the masses, the relative displacements between m_1 and m_2, $X_2 = x_2 - x_1$, and between m_2 and m_3, $X_3 = x_3 - x_2$, are negative shortly after the initiation of the load. The relative displacement, X_2, is zero at t_2' and the relative displacement, X_3, is zero at t_3'. In the time interval from 0 to t_2', the resistance, R_2, is negative; and in the time interval from 0 to t_3', R_3 is negative. The maximum elastic deflections for the first, second, and third stories are reached at time, t_{1e}, t_{2e}, and t_{3e} respectively.

The curves in figure 5.35 show three important characteristics which are generally true for any multi-story frame building. First, the ratio of the absolute displacements of any two floor masses varies over a rather large range. Hence, in design, it cannot be assumed that the absolute displacements of any two floor masses always bear a constant ratio. Secondly, the resistances, R_2, or R_3 are negative in a short interval of time after the initiation of load. The subsequent displacement of m_2 or m_3 is considerably affected by the negative portion of the resistances. Hence, in design, it is necessary to take the negative portion of the resistance function into account. Thirdly, the maximum absolute displacements, x_{1m}, x_{2m}, and x_{3m} of m_1, m_2, and m_3 do not occur at the same time. As shown in figure 5.35b, x_{1m}, x_{2m}, and x_{3m} occur at t_{1m}, t_{2m}, and t_{3m} respectively, and $t_{1m} < t_{2m} < t_{3m}$. When m_2 reaches its maximum displacement, x_{2m}, the floor mass, m_1, is moving backward due to elastic rebound. Hence, in the evaluation of the maximum relative displacement between the first and second floors, it is necessary to consider the elastic rebound of floor mass, m_1.

The floor-by-floor iteration design procedure presented in these paragraphs is intended to take these important characteristics into account. In the preliminary design of the columns for any story, the resistance as a function of time for the story must be estimated. The energy method of design given in paragraph 5-12 for single-degree dynamic systems, with slight modification can be applied to the determination of the required plastic resistance for the story. In the evaluation of the maximum allowable strain energies of the columns, the maximum absolute displacements for masses, m_1, m_2, and m_3 given in equation (5.78) must be used.

$$x_{1m} = X_{1m} \qquad (5.78a)$$

$$x_{2m} = \left(x_{1m} - X_{1e}\right) + X_{2m} = X_{1m} + X_{2m} - X_{1e} \qquad (5.78b)$$

$$x_{3m} = \left(x_{2m} - X_{2e}\right) + X_{3m} = X_{1m} + X_{2m} + X_{3m} - X_{1e} - X_{2e} \qquad (5.78c)$$

where X_{1m}, X_{2m}, and X_{3m} are the prescribed maximum relative displacements for the first, second, and third stories respectively; and X_{1e}, X_{2e} are respectively the maximum elastic relative deflections for the first and second story columns.

In equation (5.78), the maximum absolute displacements x_{1m}, x_{2m}, x_{3m} are expressed in terms of the maximum allowable relative displacements by assuming that the rebound of the lower stories equals their maximum elastic relative deflection. The maximum absolute displacements thus determined are used for the computation of the energy absorption by equations given in paragraph 5-21c.

5-21 THE PRELIMINARY DESIGN OF COLUMNS FOR ANY STORY. a. Effective Load and the Equations of Motion in Terms of Relative Displacements. In paragraph 5-19, the general procedure of iteration method of design is given. The basic principle of the preliminary design of the column for any story after all the stories below this story have been designed is given in this paragraph.

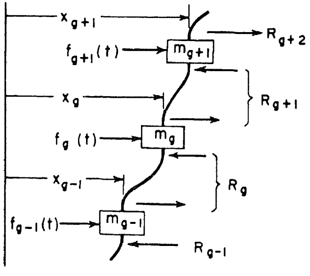


Figure 5.36. Three consecutive floors of a multi-story frame building

The equivalent dynamic system of a multi-story frame building is given in figure 5.33. Any three consecutive floors, g - 1, g, and g + 1, are shown in figure 5.36.

The equations of motion of these three floor masses are:

$$m_{g+1} \frac{d^2 x_{g+1}}{dt^2} = f_{g+1}(t) - R_{g+1} + R_{g+2} \qquad (5.79a)$$

$$m_g \frac{d^2x_g}{dt^2} = f_g(t) - R_g + R_{g + 1} \qquad (5.79b)$$

$$m_{g - 1} \frac{d^2x_{g - 1}}{dt^2} = f_{g - 1}(t) - R_{g - 1} + R_g \qquad (5.79c)$$

In the preliminary design of the columns for story g, it is assumed that all the stories above this story are absent. Neglecting $R_{g + 1}$ in equation (5.79b), the equation of motion for m_g becomes:

$$m_g \frac{d^2x_g}{dt^2} = f_g(t) - R_g \qquad (5.80)$$

This is a typical equation of motion for a single-degree dynamic system. But the design method described in paragraph 5-12 is not applicable to the determination of R_g for one reason. In paragraph 5-12, the resistance function depends on the absolute displacement of the mass supported by the columns. But in equation (5.80), although x_g is the absolute displacement, the resistance function R_g depends on the relative displacement between the floors g and $g - 1$. Hence, in the preliminary design, it is necessary to consider the relative displacement between floors as well as the absolute displacement of each floor.

The relative displacements, X_1, $X_2 \ldots X_g \ldots X_n$ are defined by the following equations:

$$\begin{aligned} X_1 &= x_1 \\ X_2 &= x_2 - x_1 \\ &\ldots\ldots\ldots \\ X_g &= x_g - x_{g - 1} \\ &\ldots\ldots\ldots \\ X_n &= x_n - x_{n - 1} \end{aligned} \qquad (5.81)$$

Substituting equation (5.81) into equations (5.79a), (5.79b), and (5.79c), the exact equations of motion in terms of relative displacements are obtained. By neglecting $R_{g + 1}$ in the preliminary design of columns for story s, the equation of motion for mass, m_g, in terms of relative displacement, X_g, is given by:

$$m_g \frac{d^2X_g}{dt^2} = f_{gX}(t) - \left(1 + \frac{m_g}{m_{g-1}}\right) R_g \qquad (5.82)$$

where

$$f_{gX}(t) = f_g(t) - \frac{m_g}{m_{g-1}} f_{g-1}(t) + \frac{m_g}{m_{g-1}} R_{g-1}$$

and $f_{gX}(t)$ is defined as the effective load for story g. The relative displacement, X_g, depends on this effective load which is a completely known function since $f_{g-1}(t)$ and $f_g(t)$ are the given loads and the resistance function, R_{g-1}, is known from the preliminary design of the columns for story $g - 1$. Once $f_{gX}(t)$ is known, the relative displacement, X_g, and the resistance, R_g, at any time can be evaluated.

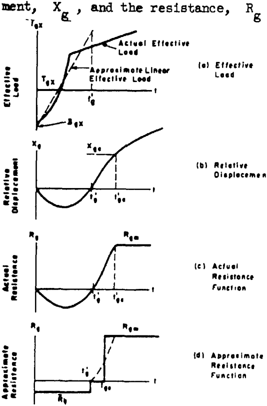


Figure 5.37. Relative displacement and resistance functions for a linear effective load, no reverse yielding

A typical shape of the effective load, $f_{gX}(t)$, is shown in figure 5.37a. The corresponding relative displacement and resistance function are given in figure 5.37b and figure 5.37c respectively. For preliminary design purposes, the actual resistance function shown in figure 5.37c is replaced by an approximate resistance function shown in figure 5.37d. The method of evaluating the parameters of the approximate resistance function, namely $\bar{R}_g$, t'_g, t_{ge}, and R_{gm} is discussed in the following paragraph.

b. Linear Effective Load. In the preliminary design of two, three, or four story frame buildings, the initial portion of the effective load for any story above the first can be approximated, with sufficient accuracy, by a linear effective load as shown in figure 5.37a. The initial portion of the actual effective load is generally always negative because the initia motion of the lower floor masses is greater than the upper floor masses. This situation is reversed as the deflections increase. The general expression of the linear effective load is:

$$f_{gX}(t) = B_{gX}\left(1 - \frac{t}{T_{gX}}\right) \qquad (5.83)$$

Note that B_{gX} is a negative quantity for reasons mentioned above. Approximate formulas for the evaluation of the parameters, B_{gX} and T_{gX}, are given in paragraph 5-23.

Substituting equation (5.83) into equation (5.82), the following equation is obtained:

$$m_g \frac{d^2X_g}{dt^2} = B_{gX}\left(1 - \frac{t}{T_{gX}}\right) - \left(1 + \frac{m_g}{m_{g-1}}\right) k_g X_g \qquad (5.84)$$

The resistance function, R_g, in equation (5.82) is replaced by $k_g X_g$ in equation (5.84).

From equation (5.84), the time, t'_g, when X_g is zero, can be evaluated. The average value of R_g, that is $\overline{R}_g$, in the time interval from 0 to t'_g, and the relative velocity dX_g/dt at t'_g also can be evaluated These quantities, evaluated on the assumption that the columns in story g remain elastic when X_g is negative, are plotted in non-dimensional form in figure 5.38.

In actual design, a trial column section is selected from which the spring constant, k_g, the maximum elastic relative deflection, X_{ge}, and the ultimate resistance, R_{gm}, can be computed. The value of T_{gX}/T'_{gn} is determined where T'_{gn} is the period of a fictitious dynamic system expressed by equation (5.84), and given by equation (5.85).

$$T'_{gn} = 2\pi\sqrt{\frac{m_g}{\left[1 + \left(m_g/m_{g-1}\right)\right]k_g}} \qquad (5.85)$$

Using T_{gX}/T'_{gn} in figure 5.38, the non-dimensional quantities:

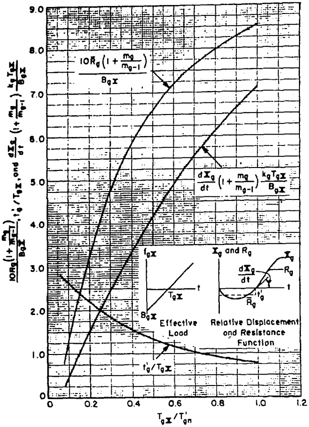


Figure 5.38. Curves for the evaluation of t_g', $\bar{R}_g$, and $\frac{dX_g}{dt}$, linear effective load

$$\frac{t_g'}{T_{gX}}\ ,\ \frac{\bar{R}_g}{B_{gX}}\left(1+\frac{m_g}{m_g-1}\right)$$

and

$$\frac{dX_g}{dt}\left(1+\frac{m_g}{m_g-1}\right)\frac{k_gT_{gX}}{B_{gX}}$$

can be determined. From these non-dimensional quantities, the values of t_g', $\bar{R}_g$, and dX_g/dt can be computed. The value of t_{ge} as indicated in figure 5.37c can be determined by assuming that the average relative velocity in the time interval from t_g' to t_{ge} is 1.5 dX_g/dt, and this is given by:

$$t_{ge} = t_g' + \frac{\frac{X_{ge}}{2}}{1.5\,\frac{dX_g}{dt}}$$

$$= t_g' + \frac{X_{ge}}{\frac{3dX_g}{dt}} \qquad (5.86)$$

Once t_g', t_{ge}, $\bar{R}_g$, and R_{gm} are determined, the approximate shape of the resistance function at any time is known and this is shown in figure 5.37d. Substituting R_g into equation (5.80), the maximum absolute displacement, x_{gm}, can be obtained by direct integration. Comparison of the computed value of x_{gm} with the design specification determines the suitability of the selected trial column section.

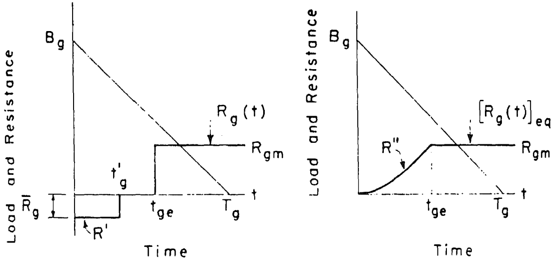
In actual application, it is rather inconvenient to carry out the integration process. In the following paragraph, a method is given for determining the suitability of the trial section based on use of the design charts presented in paragraph 5-10 for single-degree-of-freedom systems.

c. Equivalent System for Use with Design Charts. The design charts of paragraph 5-10 were constructed for the design of single-degree-of-freedom dynamic systems. These charts can be used to simplify the design of multi-story frame buildings by introducing an equivalent single-degree-of-freedom system for the g-th story.

In the previous paragraph the approximate resistance, R_g, at any time for the g-th story was developed. Suppose the external load, $f_g(t)$, is a triangular load given by:

$$f_g(t) = B_g \left(1 - \frac{t}{T_g}\right) \tag{5.87}$$

The load and resistance functions acting on m_g are shown in figure 5.39a. The resistance function does not resemble the resistance function for a single-degree-of-freedom system with zero initial displacement and velocity. Therefore it must be replaced by the resistance function, $\left[R_g(t)\right]_{eq}$ of an equivalent single-degree dynamic system shown in figure 5.39b. This is accomplished by revising the external load in a manner consistent with the change in the resistance function. For the equivalent system, the maximum



(a) Actual System

(b) Equivalent Single-Degree System

Figure 5.39. Equivalent single-degree dynamic system for the g-th story of a frame building

resistance is chosen to be equal to R_{gm} and the time when the maximum elastic deflection occurs is equal to t_{ge} . The relationship between $\left[R_g(t)\right]_{eq}$ and $R_g(t)$ is given by:

$$\left[R_g(t)\right]_{eq} = R_g(t) - R' + R'' \tag{5.88}$$

or

$$R_g(t) = \left[R_g(t)\right]_{eq} + R' - R''$$

where R' and R'' are the curves of the approximate and equivalent resistance functions respectively for times less than t_{ge} . R'' is assumed to be a parabolic curve.

Substituting equation (5.88) into equation (5.80), the following equation is obtained:

$$m_g \frac{d^2x_g}{dt^2} = f_g(t) - R_g(t) = \left[f_g(t) - R' + R''\right] - \left[R_g(t)\right]_{eq} \tag{5.89}$$

Equation (5.89) indicates that the movement of the floor mass, m_g , can be represented by the movement of the mass of an equivalent single-degree dynamic system. The external load on the equivalent system is:

$$\left[f_g(t) - R' + R''\right]$$

instead of $f_g(t)$. The work done on the equivalent system is:

$$W_g = \int \left[f_g(t) - R' + R''\right] \frac{dx}{dt}\, dt = \int_0^{t_{ge}} R'' \frac{dx}{dt}\, dt + \int \left[f_g(t) - R'\right] \frac{dx}{dt}\, dt \tag{5.90}$$

The maximum allowable strain energy of the equivalent system is:

$$\begin{aligned} E_g = \int \left[R_g(t)\right]_{eq} dx &= \int_0^{t_{ge}} R'' \frac{dx}{dt}\, dt + \int_{t_{ge}}^{t_{gm}} R_{gm} \frac{dx}{dt}\, dt \\ &= \int_0^{t_{ge}} R'' \frac{dx}{dt}\, dt + R_{gm}\left(x_{gm} - x_{ge}\right) \end{aligned} \tag{5.91}$$

where x_{ge} is the absolute displacement of m_g at time, t_{ge} .

The first term in equation (5.90) is identical with the first term in equation (5.91). Hence, in comparing the maximum work done and allowable energy absorption of the equivalent system, only the second terms in equations (5.90) and (5.91) need be considered. The method described in paragraph 5.12a can be used for the evaluation of the maximum work done. The absolute maximum work done by the load $\left[f_g(t) - R'\right]$ must be first evaluated. Next, figure 5.27 can be used to obtained the work done ratio, and the actual work done and allowable energy absorption determine the suitability of the selected trial section.

It is seen from the above discussion that, by introducing an equivalent single-degree dynamic system, the design charts given in paragraph 5-10 for single-degree dynamic systems can also be used for the design of multi-story frame buildings. The methods of evaluating the fictitious maximum work done, work done ratio, and allowable energy absorption for the equivalent system of the g-th story are described in the following three paragraphs.

Ficitious Maximum Work Done, W_{gp}. The fictitious maximum work done by the load $\left[f_g(t) - R'(t)\right]$ is given by:

$$W_{gp} = \frac{H_g^2}{2m_g} = \frac{1}{2m_g}\left[\int\left[f_g(t) - R'(t)\right]\,dt\right]^2 = \frac{1}{2m_g}\left(\frac{B_gT_g}{2} + \bar{R}_g t'_g\right)^2 \qquad (5.92)$$

For a given value of T_{gX}/T'_{gn}, the values of $\bar{R}_g$ and t'_g can be obtained from figure 5.38. Within the range of T_{gX}/T'_{gn} given by:

$$0.15 < T_{gX}/T'_{gn} < 1.0$$

the value of $\bar{R}_g t'_g$ varies from

$$0.6\,\frac{B_{gX}T_{gX}}{1 + \left(m_g/m_g - 1\right)} \quad \text{to} \quad 0.85\,\frac{B_{gX}T_{gX}}{1 + \left(m_g/m_g - 1\right)}$$

For the purpose of preliminary design, the following empirical equation may be used:

$$\bar{R}_g t'_g = 0.75\,\frac{B_{gX}T_{gX}}{1 + \left(m_g/m_g - 1\right)} \qquad (5.93)$$

Substituting the value of $\overline{R}_g t'_g$ obtained from equation (5.93) into equation (5.92), the fictitious maximum work done, W_{gp}, can be determined.

Work Done Ratio, C_{gW}. The work done ratio, $C_{gW} = W_m/W_p$ as shown in figure 5.27 depends on C_R and C_T. For the equivalent system of the g-th story, C_{gR} can be approximated from the relation:

$$C_{gR} = \frac{R_{gm}}{B_g\left(1 + \frac{\overline{R}_g t'_g}{B_g T_g}\right)} \tag{5.94}$$

but the ratio $C_{gT} = T_g/T_{gn}$ has to be evaluated before C_{gW} can be determined from figure 5.27.

A single-degree dynamic system involves three non-dimensional parameters. For the equivalent system of the g-th story, the parameters are chosen to be C_{gR}, t_{ge}/T_g, and C_gT, where t_{ge} is the time when the maximum elastic deflection, X_{ge}, occurs. The relationship between these parameters is given in figure 5.40 for the case of a triangular load. This figure can be used for the evaluation of any parameter when the other two parameters are known. For the equivalent system of a g-th story, C_{gR} is given by equation (5.94), t_{ge} is given by equation (5.86), and T_g is the given load duration. Hence, C_{gR} and t_{ge}/T_g are known. The C_{gT} ratio can be obtained from figure 5.40. The value thus obtained is defined as the equivalent value and denoted

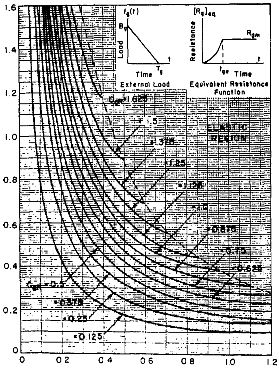


Figure 5.40. t_{ge}/T_g *curves for elasto-plastic systems, triangular load*

by $\left(C_{gT}\right)_{eq}$. Using C_{gR} and $\left(C_{gT}\right)_{eq}$, the work done ratio, C_{gW} can then be determined from figure 5.27 from which the maximum work done, W_{gm}, can be determined. From figure 5.29, the time ratio, t_{gm}/T_g is obtained, and the time, t_{gm}, when X_{gm} occurs can be determined.

Maximum Allowable Energy Absorption. The maximum allowable energy absorption as represented by the second term in equation (5.91) is:

$$E_g = R_{gm}\left(x_{gm} - x_{ge}\right) \tag{5.95}$$

The value, x_{ge}, obtained from equation (5.80) by integration, is given approximately by:

$$x_{ge} = \frac{\left(t_{ge}\right)^2}{2m_g}\left(B_g - \frac{B_g t_{ge}}{3T_g} + \overline{R}_g\right) \tag{5.96}$$

Comparison of Maximum Work Done and Maximum Allowable Energy Absorption. The value of W_{gp} is given by equation (5.92). The work done ratio, C_{gW}, can be evaluated by the method described in the preceding paragraph. The maximum allowable strain energy, E_g, is determined from equation (5.95). Comparison of the maximum work done and maximum allowable energy absorption determines the suitability of the selected trial section.

From the above discussion, it is seen that, by introducing the equivalent system for the g-th story, and by using an equivalent value, $\left(C_{gT}\right)_{eq}$, the design charts, figures 5.27, 5.28, and 5.29, which are prepared for single-degree dynamic systems, can be used for the preliminary design of the columns for a multi-story frame building. The detail procedure for design is given in paragraph 5-23.

5-22 ADJUSTMENT OF PRELIMINARY COLUMN RESISTANCE. In the design procedure outlined in paragraph 5-19, the preliminary design for each story is made assuming all higher stories are absent. That is while making a preliminary design of the g-th story, the presence of stories $g + 1$, $g + 2$, etc. is ignored. Actually, there is a force, R_{g+1}, acting on mass, m_g, from the columns of story $g + 1$ (see figure 5.36). The effect of this force may be taken into account by an adjustment of the preliminary resistance of the columns for the g-th story.

The equation of motion for m_g, when both R_g and R_{g+1} are considered, is given by equation (5.97)

$$m_g \frac{d^2x_g}{dt^2} = f_g(t) - R_g + R_{g+1} = \left[f_g(t) + R_{g+1}\right] - R_g \qquad (5.97)$$

According to equation (5.97), the resistance function, R_{g+1}, can be considered as an additional load on m_g. The approximate shapes of R_{g+1} and R_g as functions of time are shown in figure 5.41. When R_{g+1} is neglected in the preliminary design of R_g, the maximum displacement, x_{gm}, occurs at t_{gm}. The problem now is to determine by what amount R_{gm} should be increased so that when the additional load, R_{g+1} is acting on m_g, the maximum allowable displacement of m_g is still equal to x_{gm}.

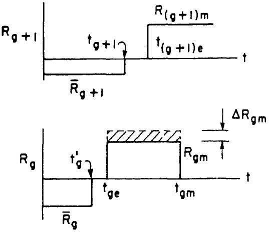


Figure 5.41. Approximate resistance functions $R_{g+1}(t)$ and $R_g(t)$

The additional load, R_{g+1}, changes both the velocity and displacement of m_g at t_{gm}. Suppose R_{gm} is changed by an amount equal to ΔR_{gm} as shown in figure 5.41; the magnitude of ΔR_{gm} should be of such value as to cancel the effect of R_{g+1} The modified value of R_{gm} is given approximately by equation (5.98).

$$\text{Modified } R_{gm} = R_{gm} + \Delta R_{gm} = R_{gm}\left[\frac{I_g + I_{g+1,\,g}}{I_g}\right] \qquad (5.98)$$

where

R_{gm} = value obtained in the preliminary design

$$I_g = R_{gm}\left(t_{gm} - t_{ge}\right) - \bar{R}_g t'_g$$

$$I_{g+1,\,g} = R_{(g+1)m}\left[t_{gm} - t_{(g+1)e}\right] - \bar{R}_{g+1} t'_{g+1}$$

5-23 COMPLETE DESIGN PROCEDURE FOR MULTI-STORY FRAME BUILDINGS. A complete design procedure for the design of multi-story frame buildings is given in this paragraph.

The procedure is set up on the assumptions that (1) the external loads can be approximated by triangular load curves, and (2) the actual effective loads can be approximated by linear effective loads. The procedure is applicable only to the design of two, three, or four story buildings. The extension of this procedure to the design of higher buildings is discussed in paragraph 5-24. Numerical examples illustrating the application of this procedure to the design of two and three story buildings are given in EM 1110-345-418.

a. Preliminary Design of the First Story Columns. In the preliminary design, the first floor is considered as a single-degree dynamic system with mass, m_1 , subjected to external load, $f_1(t)$, and all the other floors are assumed to be absent. The energy method of design given in paragraph 5-12a is used. The maximum elastic deflection, X_{1e} , the plastic resistance, R_{1m} , and the spring constant, k_1 , of the selected column section are determined. The time, t_{1m} , is determined from figure 5.29 and the time, t_{1e} , is determined from figure 5.40.

b. Preliminary Design of Columns Above the First Story. The procedure given in this section is applicable to the design of the columns of any story above the first story. Minor differences in the detail computation for different stories are noted whenever it is necessary.

Step 1. Evaluate the parameters, B_{gX} and T_{gX} , of the linear effective load according to the following approximate formulas:

For the second story:

$$B_{2X} = \left(m_2/m_1\right) B_1 - B_2 \tag{5.99}$$

$$T_{2X} = \frac{1}{\left(\frac{1}{T_2} + \frac{0.65 m_2 R_{1m}}{t_{1e} m_1 B_{2X}}\right)} \tag{5.100}$$

For any story above the second story:

$$B_{gX} = \left(m_g/m_{g-1}\right) B_{g-1} - B_g + \bar{R}_{g-1} \tag{5.101}$$

$$T_{gX} = \frac{1}{\frac{1}{T_g} + \frac{0.4m_g R_{(g-1)m}}{t_{(g-1)e} m_{g-1} B_{gX}}} \qquad (5.102)$$

where B_1, B_2, B_g are the peak values of the external loads, $f_1(t)$, $f_2(t)$, and $f_g(t)$; and, T_1, T_2, T_g are the durations of the external loads. In determining B_1, B_2, B_g, T_2, and T_g, the given external loads should be approximated by triangular load curves. Depending on which story is under consideration, the approximation should be close in the following time interval:

for the second story, $0 < t < 2t_{1e}$,
for the third story, $0 < t < 2t_{2e}$, and
for the fourth story, $0 < t < 2t_{3e}$.

In the design of buildings that are higher than two stories the value of B_{2X} may be zero or negative in which case the procedure of **paragraph** 5-24 should be used.

Step 2. Compute $\overline{R}_g t'_g$ according to equation (5.93).

$$\overline{R}_g t'_g = 0.75 \frac{B_{gX} T_{gX}}{1 + \frac{m_g}{m_{g-1}}}$$

Step 3. Compute the fictitious maximum work done, W_{gp} , according to equation (5.92).

$$W_{gp} = \frac{1}{2m_g}\left(\frac{B_g T_g}{2} + \overline{R}_g t'_g\right)^2$$

Before doing this, the given load, $f_g(t)$, is approximated by a revised triangular load curve. Depending on which story is under consideration, the approximation should be close in the following time interval:

for the second story, $0 < t < 1.3t_{1m}$,
for the third story, $0 < t < 1.3t_{2m}$,
for the fourth story, $0 < t < 1.3t_{3m}$, and generally
for the g-th story, $0 < t < 1.3t_{(g-1)m}$

The values of B_g and T_g obtained for the revised approximate triangular load will in general differ from those obtained in step 1 because they simulate the given load over a different interval of time.

Step 4. Find R_{gm}. As a first approximation, the following equations may be used.

$$\text{for any intermediate story,} \quad R_{gm} = W_{gp}/x_{gm} \tag{5.103}$$

$$\text{for the top story,} \quad R_{gm} = 1.3W_{gp}/x_{gm} \tag{5.104}$$

where x_{gm} is the maximum allowable absolute displacement for the g-th story and is given by equations similar to equation (5.78),

$$x_{1m} = X_{1m}$$

$$x_{2m} = X_{1m} + X_{2m} - X_{1e}$$

$$x_{3m} = X_{1m} + X_{2m} + X_{3m} - X_{1e} - X_{2e}$$

Step 5. Estimate the maximum elastic deflection, X_{ge}. If the heights of the first and g-th stories are equal, X_{ge} may be assumed equal to X_{1e}. Determine spring constant, k_g, from the relation,

$$k_g = R_{gm}/X_{ge} \tag{5.105}$$

Step 6. Determine T'_{gn} according to equation (5.85),

$$T'_{gn} = 2\pi \sqrt{\frac{m_g}{\left[1 + \left(m_g/m_g - 1\right)\right] k_g}}$$

Step 7. Determine T_{gX}/T'_{gn} where T_{gX} is given by equation (5.100), or equation (5.102).

Step 8. Using T_{gX}/T'_{gn} in figure 5.38, find

$$\frac{t'_g}{T_{gX}}, \quad \frac{\bar{R}_g}{B_{gX}}\left(1 + \frac{m_g}{m_g - 1}\right), \quad \text{and} \quad \frac{dX_g}{dt}\left(1 + \frac{m_g}{m_g - 1}\right)\frac{k_g T_{gX}}{B_{gX}}$$

Then compute t'_g, $\bar{R}_g$, and dX_g/dt.

Step 9. Compute t_{ge} according to equation (5.86),

$$t_{ge} = t'_g + \frac{X_{ge}}{3dX_g/dt}$$

Step 10. Compute t_{ge}/T_g and C_{gR} from equation (5.94),

$$C_{gR} = \frac{R_{gm}/B_g}{1 + \frac{\bar{R}_g t'_g}{B_g T_g}}$$

Step 11. Using t_{ge}/T_g , and C_{gR} , determine $\left(C_{gT}\right)_{eq}$ from figure 5.40.

Step 12. Using C_{gR} and $\left(C_{gT}\right)_{eq}$, determine C_{gW} and t_{gm}/T_g from figures 5.27 and 5.29 respectively. Then compute W_{gm} and t_{gm} from the relations,

$$W_{gm} = C_{gW} W_{gp} \tag{5.106}$$

$$t_{gm} = \left(t_{gm}/T_g\right) T_g \tag{5.107}$$

The value of t_{gm} is used to check whether the triangular load approximation in step 3 is acceptable. That is, if t_{gm} is much different from $1.3t_{(g-1)m}$ as assumed in step 3, a revision of the triangular load approximation may be necessary.

Step 13. Compute x_{ge} according to equation (5.96),

$$x_{ge} = \frac{\left(t_{ge}\right)^2}{2m_g}\left(B_g - \frac{B_g t_{ge}}{3T_g} + \bar{R}_g\right)$$

Step 14. Determine the maximum allowable strain energy, E_g , according to equation (5.95).

$$E_g = R_{gm}\left(x_{gm} - x_{ge}\right)$$

Step 15. Comparison of E_g and W_{gm} determines the suitability of R_{gm} . If the agreement between E_g and W_{gm} is not within 10%, the procedure from step 3 to step 14 is repeated by using a new value of R_{gm} given by,

$$R_{gm} = \frac{E_g + W_{gm}}{2\left(x_{gm} - x_{ge}\right)} \tag{5.108}$$

However, if E_g and W_{gm} agree within 10%, the value of R_{gm} obtained from equation (5.108) is approximately the maximum resistance required.

c. Adjustment of Preliminary Column Resistance R_{gm}. The effect of the $g + 1$ story columns on the movement of floor mass, m_g , is taken into account by modifying the value of R_{gm}. The modified value of R_{gm} is given by equation (5.98), as

$$\text{Modified } R_{gm} = R_{gm} \left[\left(\frac{I_g + I_{g+1,\,g}}{I_g} \right) \right]$$

where

R_{gm} = value obtained in the preliminary design

$I_g = R_{gm} \left(t_{gm} - t_{ge} \right) - \bar{R}_g t'_g$

$I_{g+1,\,g} = R_{(g+1)m} \left[t_{gm} - t_{(g+1)e} \right] - \bar{R}_{g+1} t'_{g+1}$

Starting from the top story, the plastic resistances of different stories should be modified one after the other. In the modification of R_{gm} by equation (5.98), the value of $R_{(g+1)m}$ should be the modified value rather than the value obtained in the preliminary design. In the computations of I_1 for the bottom story, the value of $\bar{R}_1$ is zero and the value of t_{ge} is equal to $2t_{1e}/3$.

d. Selection of Final Column Sections. The maximum horizontal resistance of the selected columns for the g-th story, including the effect of vertical loads acting on the columns as outlined below, should be close to the modified value of R_{gm} obtained in the previous paragraph. In determining the effect of vertical load, its average value in the time interval,

$$t_{ge} < t < t_{gm}$$

should be used. The expression of the ultimate resistance of the selected columns is given by:

$$R_{gm} = \frac{2nM_D}{h_g} - \frac{\left(\Sigma P_v \right) X_{gm}}{2h_g} \tag{5.109}$$

where

M_D = plastic resisting moment of the selected section after the effect of the vertical load is taken into account.

ΣP_v = the time average of the total vertical load in the time interval, $t_{ge} < t < t_{gm}$.

n = number of columns in the g-th story.

h_g = height of the g-th story.

After the column section has been selected, the spring constant, k_g, and the limiting elastic deflection, X_{ge}, are determined. In the evaluation of these parameters, the effect of girder flexibility may be approximately accounted for (see EM 1110-345-417 and EM 1110-345-418).

e. Numerical Analysis of the Entire Building. A numerical analysis for the entire building should be carried out to determine the suitability of the selected columns. If the result of the numerical analysis indicates that the maximum absolute displacement, x_{gm}, is greatly different from the design specification, all the columns in and below the g-th story should be revised. The revised value of R_{gm} can be determined by a trial and error method or from the relation,

$$R_{gm} = \frac{R'_{gm}\left(x'_{gm} - x'_{ge}\right) - R'_{(g+1)m}\left(x'_{gm} - x'_{ge}\right)}{x_{gm} - x'_{ge}} + \frac{R_{(g+1)m}\left(x_{gm} - x'_{ge}\right) - f_g\left(t'_{gm}\right)\left(x'_{gm} - x_{gm}\right)}{x_{gm} - x'_{ge}} \qquad (5.110)$$

where R'_{gm} and $R'_{(g+1)m}$ are the average values of the maximum resistance used in the numerical analysis over the time interval, $t_{ge} < t < t_{gm}$; and x'_{gm} and x'_{ge} are the displacements obtained from the numerical analysis. $R_{(g+1)m}$ is the revised maximum resistance of story $g + 1$; x_{gm} is the prescribed maximum allowable absolute displacement; and $f_g\left(t'_{gm}\right)$ is the magnitude of the external load, $f_g(t)$, at the time when x'_{gm} occurs.

New column sections should be selected according to the revised values of the plastic resistances. The spring constants and the maximum elastic deflection should then be determined. The numerical analysis should be repeated using the revised column sections.

5-24 PROCEDURE FOR BUILDINGS OF MORE THAN TWO STORIES. The design procedure given in paragraph 5-23 is for the case of a linear effective load which is applicable only to buildings two stories or less in height. In the design of higher buildings (higher than two stories) the value of B_{2X} computed from equation (5.99) may be zero or negative. The effective load, $f_{2X}(t)$, and the relative displacement, X_2, are then positive immediately after time, $t = 0$. Thus when B_{2X} computed from equation (5.99) is zero or negative, the time, t_{2e}, can be taken to be $5t_{1e}/3$. Similarly, if B_{3X} computed from equation (5.101) is zero or negative, the value of t_{3e} can be taken to be $\left(t_{2e} + \frac{2t_{1e}}{3}\right)$. For buildings of more than two stories, if B_{gX} is close to zero or negative, $\overline{R}_g$ should be considered equal to zero and the procedures from step 5 to step 9 in paragraph 5-23b should be omitted in the preliminary design of the second or third story columns.

5-25 ACCURACY OF FLOOR-BY-FLOOR DESIGN METHOD. A number of assumptions have been used in this design method. First is the replacement of actual effective load by a linear load in which the parameters, B_{gX} and T_{gX} are determined by approximate formulas. Second, approximate formulas are used in the determination of t_{ge} and C_{gR}. Third, in the evaluation of strain energies and work done, the portion of $R''(t)$ shown in figure 5.39b is neglected. And fourth, in the determination of the plastic resistance, R_m, of the chosen column section from equation (5.109), the time average of the total vertical load on the columns is used.

There is error involved in each assumption. The accumulative error for an actual design problem may be quite large. However, based on a number of examples which have been worked covering a wide range of buildings under various loadings, the maximum resistance of the columns for any story obtained by this floor-by-floor iteration method has been found to be generally within 20% of the desired value, for cases in which the prescribed maximum allowable relative displacement was at least three times larger than the maximum relative elastic deflection.

For multi-story frame buildings, a small change in the plastic resistance generally results in a large change in the relative displacement. Although the result of the numerical analysis may indicate a large error in the relative displacement, this usually represents small error in the plastic resistance. Since the plastic resistance cannot be evaluated exactly, accuracy greater than 20% in the design of columns on the basis of relative displacement is not warranted. An error of 20% in the relative displacement may correspond to an error of only 5% in the plastic resistance.

DESIGN OF MULTI-STORY BUILDINGS FOR ELASTIC DEFORMATION

5-26 INTRODUCTION. In the design of a multi-story building for elastic deformation the maximum allowable lateral deflection of the columns, under dynamic load, should not exceed that which causes yielding. In the equivalent system for frame buildings as shown in figure 5.33, the moment in the columns of the g-th story is proportional to the relative displacement X_g. Direct stresses are developed in the columns due to over-all bending but these may be ignored in buildings of three or four stories. In frame buildings, where the direct stress due to over-all bending is appreciable, the preliminary design could still be made without including the effect of the direct stress. This would be considered, however, in selecting the column sections. Hence, the elastic design of frame buildings for elastic deformation can be based on the requirement that the maximum allowable relative displacement for any story must be smaller than the relative elastic deflection at the yield point stress.

However, in the equivalent system of a shear wall building as shown in figure 5.32, the stresses in the shear wall at any story are a function of the over-all bending and shear in the wall and hence depend on the relative displacements of all stories rather than just the relative displacement of adjacent stories. The use of absolute displacements is desirable in the design of shear walls because it is the simplest method to express the relative displacement between all floors in a shear wall building, whether the floors are adjacent or non-adjacent.

The characteristics of a multi-story building designed for plastic deformation discussed in paragraph 5-20 are also generally true when designed for elastic deformation. These characteristics are: (1) the relative displacements in the upper stories are negative shortly after the initiation of the load; and (2) the elastic rebound of floor masses must be considered in the evaluation of the relative displacements. In design for plastic deformation, the resistance, R_g, is assumed to be a constant after the elastic limit is reached. The design of the g-th story columns is reduced to the determination of the shape R_g as shown in figure 5.37d. Moreover, the elastic rebound of floor masses is considerably smaller than the allowable relative displacement. Hence, the use of approximate formula in estimating the magnitude of elastic rebound is permissible. But in an elastic system, the resistance, R_g, always varies with time. The elastic rebound is about of the same order of magnitude as the relative displacement, and approximate evaluation of the elastic rebound is not permissible. Because of these differences, the elastic design of multi-story buildings is more involved than the design for plastic deformation. In most cases, a trial and error design procedure may have to be used. The elastic design of multi-story frame and shear wall buildings is briefly discussed in paragraphs 5-27 and 5-28 respectively. The design of shear wall buildings is more fully discussed in EM 1110-345-419.

5-27 DESIGN OF MULTI-STORY FRAME BUILDINGS FOR ELASTIC DEFORMATION. One possible approach to the elastic design of multi-story buildings is the preparation of non-dimensional graphs. The feasibility of this method is illustrated by the characteristics of a two-story frame building. The equivalent dynamic system of the building is shown in figure 5.42. The external loads, $f_1(t)$ and $f_2(t)$ are assumed to be triangular with load durations T_1 and T_2 respectively. The relative displacements, X_{1s} and X_{2s} when the peak external loads are applied statically, are given by:

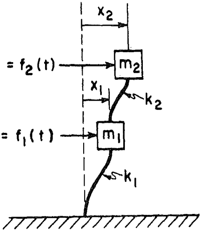


Figure 5.42. Equivalent dynamic system for two-story building

$$X_{2s} = \frac{B_2}{2k_2} \qquad (5.111)$$

$$X_{1s} = \frac{B_1 + B_2}{2k_1} \qquad (5.112)$$

Suppose X_{2m} and X_{1m} denote the maximum relative displacements under dynamic load; the dynamic load factors for the second and first stories are defined by the following relations:

$$(D.L.F.)_2 = X_{2m}/X_{2s} \qquad (5.113)$$

$$(\text{D.L.F.})_1 = X_{1m}/X_{1s} \tag{5.114}$$

These dynamic load factors depend on the non-dimensional parameters k_2/k_1, T_2/T_1, B_2/B_1, and T_1/T_{1n}, where T_{1n} is equal to $2\pi\sqrt{m_1/k_1}$. The general shape of the dynamic load factor curves is shown in figure 5.43 for specific values of k_2/k_1, T_2/T_1, and B_2/B_1. Before these curves can be used for practical design, a series of similar graphs must be prepared for different combinations of m_2/m_1, B_2/B_1, and T_2/T_1 to cover the probable range of practical buildings. For practical cases, it is believed the range of m_2/m_1, B_2/B_1, T_2/T_1, and k_2/k_1 would vary within narrow limits. It is even possible that they might be assumed constant for certain types of construction. Suppose four different values are used for each non-dimensional parameter, then the total number of graphs needed is 64. By similar considerations for a three-story building, the total number of graphs required is 2401. Since such a large number of graphs are needed, it appears impractical in general to prepare non-dimensional graphs for use in the design of multi-story dynamic systems.

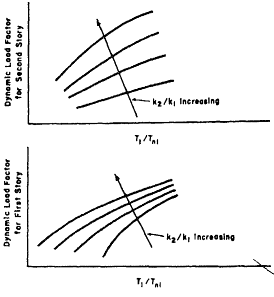


Figure 5.43. Dynamic load factors for two-story building for elastic design, m_2/m_1 = constant, T_2/T_1 = constant, B_2/B_1 = constant

For practical design, a trial and error procedure may have to be used. A set of columns for the entire building is selected. The suitability of the selected columns is determined by the result of a numerical analysis. In the preliminary selection of columns for the g-th story, an equivalent single-degree dynamic system, shown in figure 5.44, may be used. In the equivalent system, all the floor masses and external loads above the g-th story are assumed to be concentrated at the g-th floor. The

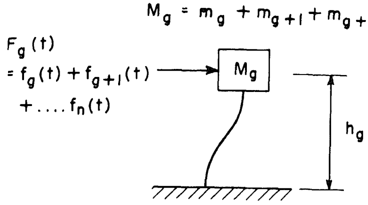


Figure 5.44. Equivalent single-degree system for the g-th story

design method on the basis of dynamic load factor given in paragraph 5-12c is used for the determination of the columns for the g-th story. The system shown in figure 5.44 is used for the preliminary design of columns in the first, second, up to (n - 1)th story. For the top story, the external load is assumed to be $2f_n(t)$ instead of $f_n(t)$.

Because of higher modes of oscillation in multi-story buildings, a large number of maximum and minimum values are presented when the relative displacements are plotted versus time. Hence, in the numerical analysis, the integration process should be carried out far enough to insure that the absolute maximum relative displacements for all stories are obtained.

5-28 DESIGN OF MULTI-STORY SHEAR WALL BUILDINGS FOR ELASTIC DEFORMATION.

In the elastic design of shear wall buildings, the controlling factor may be shear and/or bending stresses. For the evaluation of both stresses, the building is replaced by a multi-degree dynamic system discussed in paragraph 5-16a. The method of evaluating the parameters of the dynamic system, the detailed preliminary design procedure, a discussion of design techniques for both elastic and plastic action, and numerical examples of shear wall building design are given in EM 1110-345-419.

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APPENDIX A

EFFECT OF VARIATION OF PARAMETERS ON THE RESPONSE OF SINGLE-DEGREE DYNAMIC SYSTEMS

A-01 INTRODUCTION. In paragraph 5-10, the response of single-degree dynamic systems subjected to simplified loads is presented in non-dimensional graphs. For a given problem, when the peak load B, the load duration T, the mass m, the spring constant k and the plastic resistance R_m are all known, the maximum work done, W_m, the maximum displacement, x_m, and the time, t_m, when x_m occurs can be determined from these non-dimensional graphs. For the convenience of discussion, the first five variables (B, T, m, k, and R_m) are considered as independent variables in this appendix.

For design applications, in addition to these non-dimensional graphs, it would be helpful to know how the response of a dynamic system varies with the shape, the peak value, and the duration of the load pulse, and how the response is affected by the mass, the spring constant, and the plastic resistance. In other words, it is of interest to know the effects of small increments of the independent variables on the response. Two practical design problems, in which this is of importance, are the following:

The first problem is concerned with uncertainties. Since all the independent variables are subject to uncertainties in their determination, it would be of interest to know what is the effect of these uncertainties on the response and what is the necessary accuracy in the determination of the variables such that the response of a dynamics system is within a given tolerable limit.

The second problem occurs when it is intended to answer the following two questions which often arise at the completion of a trial design computation. These questions are: (1) What is the permissible difference between the specified maximum displacement and the corresponding computed value of a given design; and (2) if a trial design is judged to be unsatisfactory, by what amounts should the variables be changed in the next trial design. For numerical examples, suppose for a given system a 10% change in the mass results in 1% change in the maximum displacement, then an accurate determination of the mass would not be necessary. On the other hand; if a 10% change in the plastic resistance results in a 50% change in x_m, a close agreement between the specified maximum displacement and the corresponding computed value would not be warranted since the determination of the plastic resistance may be subject to an uncertainty greater than 10%.

The effects of small increments of the independent variables on the response are discussed in this appendix. These effects are expressed in terms of percentage-increment ratios which are non-dimensional quantities and are defined in paragraph A-02. In paragraph A-03, the technique of non-dimensional analysis is applied in deriving certain functional relationships among the percentage-increment ratios. Because of these relationships, the effect of small increments of all of the variables (B, T, m, k, and R_m) can be condensed into two sets of curves. The percentage-increment ratio for linearly elastic, completely plastic, and elasto-plastic systems subjected to both rectangular and triangular loads are discussed in paragraphs A-04, A-05, and A-06 respectively. The effect of load shape on the response is discussed in paragraph A-07. The expressions of various percentage-increment ratios for linearly elastic, completely plastic, and elasto-plastic systems subjected to rectangular and triangular loads are summarized in paragraph A-08.

It is to be noted that all of the material in this appendix is derived from the corresponding material in EM 1110-345-415 through a process of differentiation. The percentage-increment ratios are related to the partial derivatives, (for example, $\frac{\partial x_m}{\partial T}$, $\frac{\partial x_m}{\partial k}$); and the graphs in this appendix are related to the slopes of the corresponding graphs in paragraph 5-10. Although these two sets of graphs are derivable from each other, they are intended for different applications. Generally speaking, for routine design computation, it would be easier and the result more accurate when the graphs in paragraph 5-10 are used, while for those special applications discussed earlier in this paragraph, it would be better to use the graphs in this appendix. Beside these special applications, the graphs of percentage-increment ratios also illustrate many important properties of structures under dynamic loads. For this reason, although quantitative data are presented in figures A.1 through A.8, the discussion in paragraphs A-04, A-05, and A-06 is more or less qualitative in nature. This quantitative discussion should then serve as a rule of thumb for a designer to make proper judgement on the many practical problems which are not necessarily related to any routine design computation.

A-02 SENSITIVITY AND PERCENTAGE-INCREMENT RATIO. a. Sensitivity. The maximum displacement, x_m, generally depends on all the variables, B, T, m, k, and R_m. Suppose for the time being only the load duration, T, is varying. Then x_m is a function of T. If the load duration is changed from T to $T + \Delta T$, the corresponding maximum displacement is changed from x_m to $x_m + \Delta x_m$. For small increments, Δx_m and ΔT are related by equation (A.1):

$$\Delta x_m = \frac{\partial x_m}{\partial T} \Delta T \tag{A.1}$$

or

$$S_{xT} = \frac{\Delta x_m}{\Delta T} = \frac{\partial x_m}{\partial T} \tag{A.2}$$

where

ΔT = the increment of the load duration.

Δx_m = the corresponding increment of the maximum displacement.

$\frac{\partial x_m}{\partial T}$ = a partial derivative.

The quantity, S_{xT}, being a partial derivative, is defined as the sensitivity of the maximum displacement with respect to the load duration. Thus the sensitivity S_{xT} is equal to the ratio of two increments Δx_m and ΔT and is a dimensional quantity: For example, when S_{xT} is equal to 0.01 inch/sec, this indicates that if the load duration is increased by 0.1 seconds, the maximum displacement would be increased by 0.001 inches provided the increments are small in comparison with their original values.

Although the dimensional quantity, sensitivity, is very useful in numerical computation, it does not express clearly the physical significance of "sensitivity." From the numerical magnitude of the sensitivity, it is difficult to perceive whether the maximum displacement is "very sensitive" or "not sensitive" to any variation of the load duration. This is because the original values x_m and T to which Δx_m and ΔT are to be added are not included in the definition of the sensitivity. Hence in this appendix, instead of using increments and sensitivities, the non-dimensional quantities, percentage-increments and percentage-increment ratios are used. These non-dimensional quantities are defined in paragraph A-02b.

b. Percentage-Increment Ratios. If the left hand side of equation (A.1) is divided by x_m and the right hand side is divided by $\frac{x_m}{T} T$, the result is given by equation (A.3).

$$\frac{\Delta x_m}{x_m} = \left(\frac{T}{x_m} \frac{\partial x_m}{\partial T}\right)\frac{\Delta T}{T}$$

$$= I_{xT} \frac{\Delta T}{T} \tag{A.3}$$

where

$\frac{\Delta x_m}{x_m}$ = percentage-increment in x_m

$\frac{\Delta T}{T}$ = percentage-increment in T

and

$$I_{xT} = \frac{T}{x_m} \frac{\partial x_m}{\partial T}$$

= The ratio of the percentage change in x_m to the percentage change in T while all the remaining independent variables are held constant.

The quantity I_{xT}, with two subscripts, is defined as the percentage-increment ratio.

In the discussion so far, x_m and T are considered respectively as the dependent and independent variables. By a similar procedure, if the variables are the maximum work done and the peak load B, the percentage-increment ratio I_{WB} would give the ratio of the percentage-increment of W_m to the corresponding percentage-increment of the peak load B.

There are a number of percentage-increment ratios for a single-degree dynamic system subjected to simplified loads. These ratios are described in paragraph A-02c.

c. Various Percentage-Increment Ratios for Single-Degree Dynamic Systems. For a single-degree dynamic system subjected to simplified loads there are five independent variables, B, T, m, k, and R_m. When any one of these variables is changed, the maximum displacement, in general, is affected. The total effect on x_m when all the variables are changed can be obtained approximately by the method of superposition. Thus the total increment Δx, for the general case instead of being given by equation (A.1), is given by:

$$\Delta x_m = \frac{\partial x_m}{\partial T}\Delta T + \frac{\partial x_m}{\partial m}\Delta m + \frac{\partial x_m}{\partial k}\Delta k + \frac{\partial x_m}{\partial B}\Delta B + \frac{\partial x_m}{\partial R_m}\Delta R_m \qquad (A.4)$$

According to equation (A.2), the different partial derivatives in equation (A.4) can be defined as different sensitivities.

Applying the technique used in deriving equation (A.3), the general expression for the percentage-increment in the maximum displacement can be derived and this is given by equation (A.5).

$$\frac{\Delta x_m}{x_m} = I_{xT}\frac{\Delta T}{T} + I_{xm}\frac{\Delta m}{m} + I_{xk}\frac{\Delta k}{k} + I_{xB}\frac{\Delta B}{B} + I_{xR}\frac{\Delta R_m}{R_m} \qquad (A.5)$$

By the same method of derivation, if W_m instead of x_m is considered as the independent variable, the general expression of the percentage-increment of the maximum work done is given by equation (A.6):

$$\frac{\Delta W_m}{W_m} = I_{WT}\frac{\Delta T}{T} + I_{Wm}\frac{\Delta m}{m} + I_{Wk}\frac{\Delta k}{k} + I_{WB}\frac{\Delta B}{B} + I_{WR}\frac{\Delta R_m}{R_m} \qquad (A.6)$$

The quantities, I_{xT}, I_{xm}, I_{xk}, I_{xB}, and I_{xR} in equation (A.5) are the percentage-increment ratios for the maximum displacement, while the quantities, I_{WT}, I_{Wm}, I_{Wk}, I_{WB} and I_{WR} in equation (A.6) are the percentage-increment ratios for the maximum work done.

The percentage-increment ratio gives quantitatively the effect of a given variable on the response of the dynamic system. For example if the numerical values of I_{xT} for different structures are different, the larger the value of I_{xT} is, the more the maximum displacement will be affected by any change in the load duration. Similarly, for a given structure if I_{xR} is numerically larger than I_{xm}, the maximum displacement will be affected more by the plastic resistance than by the mass, m . On the other hand, if I_{xk} is numerically small, say less than 0.2, the maximum displacement is practically a constant when the spring constant k varies.

Equations (A.5) and (A.6) are derived by partial differentiation and superposition. Hence, in the evaluation of the percentage change in x_m or W_m for a given structure dynamic system, these simplified equations, in general, are applicable only when the increments in B, T, m, k, or R_m are not more than 10%.

In equations (A.5) and (A.6), there are ten percentage-increment ratios. These ratios are not all independent. The relationships among them are derived in paragraph A-03.

A-03 RELATIONSHIPS AMONG THE PERCENTAGE-INCREMENT RATIOS. a. Increment Ratio for Non-Dimensional Parameters. The discussion in paragraph 5-10 shows that only three non-dimensional parameters are needed to express the functional relationships of all the variables of an elasto-plastic system subjected to simplified loads. For example in figures 5.25 and 5.28, the three parameters are, $\frac{x_m}{x_e}$, $\frac{T}{T_n}$ and $\frac{R_m}{B}$, which are denoted by C_x, C_T, and C_R respectively. That is

$$C_x = \frac{x_m}{x_e} \tag{A.7}$$

$$C_T = \frac{T}{T_n} = \frac{T}{2\pi}\sqrt{\frac{k}{m}} \tag{A.8}$$

$$C_R = \frac{R_m}{B} \tag{A.9}$$

In terms of these parameters, the dependent variable of the dynamic system is C_x instead of x_m, and the independent variables are C_T and C_R instead of B, T, m, k and R_m . For small increments in C_T and C_R, the corresponding increment C_x is given by equation (A.10).

$$\Delta C_x = \frac{\partial C_x}{\partial C_T}\Delta C_T + \frac{\partial C_x}{\partial C_R}\Delta C_R \tag{A.10}$$

where ΔC_x, ΔC_T and ΔC_R are small increments, $\frac{\partial C_x}{\partial C_T}$ and $\frac{\partial C_x}{\partial C_R}$ are partial derivatives.

Applying the same technique used in the derivation of equation (A.3) and (A.5), the percentage-increment of C_x, can be derived and is given by equation (A.11)

$$\frac{\Delta C_x}{C_x} = \left(\frac{C_T}{C_x}\frac{\partial C_x}{\partial C_T}\right)\frac{\Delta C_T}{C_T} + \left(\frac{C_R}{C_x}\frac{\partial C_x}{\partial C_R}\right)\frac{\Delta C_R}{C_R} = I_1 \frac{\Delta C_T}{C_T} + I_2 \frac{\Delta C_R}{C_R} \qquad (A.11)$$

where

$\frac{\Delta C_x}{C_x}$ = percentage-increment in C_x

$\frac{\Delta C_T}{C_T}$ = percentage-increment in C_T

$\frac{\Delta C_R}{C_R}$ = percentage-increment in C_R

$I_1 = \frac{C_T}{C_x}\frac{\partial C_x}{\partial C_T}$

= the percentage-increment ratio of C_x with respect to C_T

$I_2 = \frac{C_R}{C_x}\frac{\partial C_x}{\partial C_R}$

= the percentage-increment ratio of C_x with respect to C_R.

There are only two percentage-increment ratios in equation (A.11). It can be concluded that of all the percentage-increment ratios of an elasto-plastic system, only two ratios are independent. Hence the ten percentage-increment ratios in equations (A.5) and (A.6) can be expressed in terms of I_1 and I_2.

This conclusion can also be deducted by physical reasoning. C_x is a function of C_R and C_T. If for a given case C_R is a constant, C_x would remain a constant provided that C_T is unchanged although T, m and/or k may be varying. This indicates that I_{xT}, I_{xm}, and I_{xk} can be expressed in terms of I_1. Similarly, it can be concluded that I_{xB} and I_{xR} can be expressed in terms of I_2. The equations expressing those relationships are given in paragraph A-03b.

b. Percentage-Increment Ratios in Terms of I_1 and I_2 Substituting $\frac{R_m}{k}$ for x_e in equation (A.7), the following equation is obtained:

$$C_x = \frac{x_m k}{R_m} \tag{A.12}$$

Taking logarithms of both sides of this equation, then differentiating, and then replacing the differential notation "d" by the increment notation "Δ," equation (A.13) is obtained.

$$\frac{\Delta C_x}{C_x} = \frac{\Delta x_m}{x_m} + \frac{\Delta k}{k} - \frac{\Delta R_m}{R_m} \tag{A.13}$$

Similarly, from equations (A.8) and (A.9), the following two equations are obtained respectively:

$$\frac{\Delta C_T}{C_T} = \frac{\Delta T}{T} + \frac{1}{2}\frac{\Delta k}{k} - \frac{1}{2}\frac{\Delta m}{m} \tag{A.14}$$

$$\frac{\Delta C_R}{C_R} = \frac{\Delta R_m}{R_m} - \frac{\Delta B}{B} \tag{A.15}$$

substituting equations (A.13), (A.14) and (A.15) into equation (A.11) and rearranging:

$$\frac{\Delta x_m}{x_m} = I_1 \frac{\Delta T}{T} - \frac{I_1}{2}\frac{\Delta m}{m} - \left(1 - \frac{I_1}{2}\right)\frac{\Delta k}{k} - I_2 \frac{\Delta B}{B} + \left(1 + I_2\right)\frac{\Delta R_m}{R_m} \tag{A.16}$$

Comparing equation (A.16) with equation (A.5), the following relationships among the percentage-increment ratios are obtained:

$$I_1 = I_{xT} = - 2\ I_{xm} = 2\left(I_{xk} + 1\right) \tag{A.17}$$

$$I_2 = - I_{xB} = \left(I_{xR} - 1\right) \tag{A.18}$$

So far in the derivation, x_m is considered as the dependent variable. By the same method, the percentage-increment ratio when W_m is the dependent variable can be derived. For an elasto-plastic system, referring to equation (5.38), the expression for W_m is given by

$$W_m = R_m\left(x_m - \frac{x_e}{2}\right) \qquad (A.19)$$

Taking logarithms of both sides, differentiating and rearranging:

$$\frac{\Delta W_m}{W_m} = \frac{2C_x}{2C_x - 1}\frac{\Delta x_m}{x_m} + \frac{2\left(C_x - 1\right)}{2C_x - 1}\frac{\Delta R_m}{R_m} + \frac{1}{2C_x - 1}\frac{\Delta k}{k}. \qquad (A.20)$$

Substituting equation (A.16) into (A.20), the expression of $\frac{\Delta W_m}{W_m}$ in terms of the percentage changes of the dependent variables is obtained:

$$\frac{\Delta W_m}{W_m} = \frac{2C_x}{2C_x - 1} I_1 \frac{\Delta T}{T} - \frac{C_x}{C_x - 1} I_1 \frac{\Delta m}{m} - \left(1 - \frac{C_x}{2C_x - 1} I_1\right)\frac{\Delta k}{k} - \frac{2C_x}{2C_x - 1} I_2 \frac{\Delta B}{B} + \left(2 + \frac{2C_x}{2C_x - 1} I_2\right)\frac{\Delta R_m}{R_m} \qquad (A.21)$$

Comparing equation (A.21) with equation (A.6), the relationships among the percentage-increment ratios of W_m are obtained and these are given by equations (A.22) and (A.23).

$$I_1 = \frac{2C_x - 1}{2C_x} I_{WT} = \frac{1 - 2C_x}{C_x} I_{Wm} - \frac{2C_x - 1}{C_x}\left(I_{Wk} + 1\right) \qquad (A.22)$$

$$I_2 = \frac{1 - 2C_x}{2C_x} I_{WB} = \frac{2C_x - 1}{2C_x}\left(I_{WR} - 2\right) \qquad (A.23)$$

Equations (A.20) through (A.23) are derived on the basis that W_m is given by equation (A.19), which is true for an elasto-plastic system. It can be shown that these equations are also applicable for both linearly elastic and completely plastic systems provided that C_x is replaced by unity for the former while the limiting value with C_x approaching infinity is used for the latter.

Equations (A.17) and (A.22) show that I_{xT}, I_{xm}, I_{xk}, I_{WT}, I_{Wm} and I_{Wk} can be expressed in terms of I_1, while equations (A.18) and (A.23) show that I_{xB}, I_{xR}, I_{WB} and I_{WR} can be expressed in terms of I_2. Therefore, the graphical presentation of the percentage-increment ratios is greatly simplified by these functional relationships. When any one of the percentage-increment ratios in the first group is computed, all the remaining I's are also known. The same is true for the second group of I's . Thus only two sets of curves are needed for the complete graphical presentation of all the I's of an elasto-plastic system subjected to simplified loads. In this appendix, one set of curves is for I_1 , which is also equal to I_{xT} . The other set is for I_{xR} which is equal to $\left(1 + I_2\right)$. These curves are described in paragraph A-06.

c. Other Percentage-Increment Ratios. The discussion so far covers the cases when the independent variables are B, T, m, k, and R_m; and the dependent variables are x_m and W_m . For special application, the percentage-increment ratio for variables other than those mentioned above may be needed. Because of the non-dimensional nature of the percentage-increment ratio, the expressions for additional percentage-increment ratios can be derived from those already presented by a method of superposition. A few examples are given for illustration.

As a first example, suppose the effect of the limiting elastic displacement x_e on x_m is to be considered. Since

$$x_e = \frac{R_m}{k}$$

therefore

$$\frac{\Delta x_e}{x_e} = \frac{\Delta R_m}{R_m} - \frac{\Delta k}{k} \tag{A.24}$$

The percentage-increment ratio of x_m with respect to x_e is then given by:

$$I_{xe} = I_{xR} - I_{xk} \tag{A.25}$$

Where I_{xe} is ratio of the percentage change in x_m to the corresponding percentage change in x_e .

As a second example, the load pulse H , which is the area under the load time curve as defined by equation (5.5), is considered as an independent variable. For rectangular or triangular load, the load pulse is proportional to the product of B and T. Then equation (A.26) is obtained:

$$I_{xH} = I_{xB} + I_{xT} \tag{A.26}$$

where I_{xH} is the percentage-increment ratio of the maximum displacement x_m with respect to the load pulse H.

As a third example, the effect of $\frac{B}{T}$ on the maximum displacement is considered. For a given triangular or rectangular load, this variable is the ratio of the two sides of the triangle or rectangle. Hence in a very restricted sense, it represents the "shape" or "proportion" of the load curve with a given geometry. Let this variable be denoted by J, that is:

$$J = \frac{B}{T} \tag{A.27}$$

The percentage-increment ratio I_{xJ} is given by:

$$I_{xJ} = I_{xB} - I_{xT} \tag{A.28}$$

Where I_{xJ} is the ratio of the percentage change in x_m to the corresponding percentage change in $\frac{B}{T}$.

A-04 PERCENTAGE-INCREMENT RATIOS FOR LINEARLY ELASTIC SYSTEMS. The percentage-increment ratios for the maximum displacements of linearly elastic systems subjected to triangular or rectangular loads are given in this paragraph. In deriving the expressions for different percentage-increment ratios, the methods described in paragraphs 5-07b, A-02 and A-03 are used. For example, the case of a triangular load with $\frac{T}{T_n} > 0.371$ is considered. This corresponds to $t_m < T$. The expression for x_m for this case is given by equation (A.29).

$$x_m = \frac{B}{k}\left[1 - \frac{\tan^{-1}\left(\frac{2\pi T}{T_n}\right)}{\pi \frac{T}{T_n}}\right] \tag{A.29}$$

The expressions for I_{xB} and I_{xT} derived from this equation are respectively given by the following two equations:

$$I_{xB} = 1.0 \tag{A.30}$$

$$I_{xT} = \frac{2\pi\left[\dfrac{4\pi^2 C_T^{\ 2}}{1 + 4\pi^2 C_T^{\ 2}}\right]}{2\pi - \dfrac{\tan^{-1} 2\pi C_T}{C_T}} - 1 \tag{A.31}$$

Applying equation (A.17), the corresponding expression for I_{xm} and I_{xk} can be derived from equation (A.31).

Similar expressions for all the percentage-increment ratios for triangular loads with $\frac{T}{T_n} < 0.371$, and also for rectangular loads can be obtained. These expressions are all summarized in paragraph A-08a.

The percentage-increment ratios for linearly elastic systems are plotted in figures A.1 and A.2. The first figure is for rectangular load while the second is for triangular load. From these two figures, the following conclusions can be drawn:

(1) For both types of load, I_{xB} is a constant and equal to unity. The maximum displacement of a linearly elastic system is always directly proportional to the peak load B .

(2) When C_T is less than 0.2, I_{xT} is approximately constant and equals unity for both types of loads. Thus for short duration load, the maximum displacement of a linearly elastic system is directly proportional to the load duration T . On the other hand, for a rectangular load, I_{xT} is zero when $C_T > 0.5$ and for a triangular load, I_{xT} is less than 0.1 when $C_T > 3.0$. Therefore the maximum displacement is practically independent of the load duration for long duration load.

(3) For both types of loads, when $C_T < 0.2$, both I_{xm} and I_{xk} are approximately equal to -0.5. Thus for rectangular or triangular loads of short duration the maximum displacement is inversely proportional to both the square root of the mass m and the square root of the spring constant k . On the other hand, for a rectangular load when $C_T > 0.5$, I_{xm} equals zero and I_{xk} equals -1.0. These numerical values are also approximately correct for a triangular load when $C_T > 3.0$. Thus for long duration triangular or rectangular load, the maximum displacement of a linear elastic system is practically independent of the mass but is inversely proportional to the spring constant k .

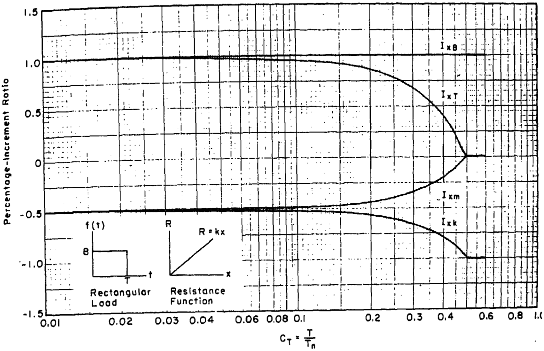

Figure A.1. Percentage increment ratios for a linearly elastic system subjected to rectangular loads

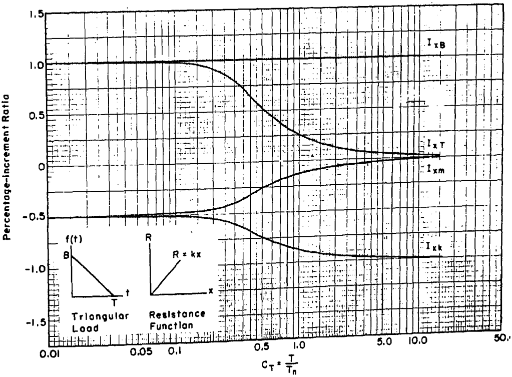

Figure A.2. Percentage increment ratios for a linearly elastic system subjected to triangular loads

(4) The numerical values of I_{xB}, I_{xT}, I_{xm} and I_{xk} are always less than unity. Thus the reliability of the maximum displacement determined by a design computation is better than or as good as the uncertainties in the determination of any variable.

A-05 PERCENTAGE-INCREMENT RATIOS FOR COMPLETELY PLASTIC SYSTEMS. Applying the method described in paragraph 5-07c, the expression of x_m for a completely plastic system subjected to rectangular loads can be obtained and is given by equation (A.32).

$$x_m = \frac{1}{2} \frac{T^2}{m} \frac{B\left(B - R_m\right)}{R_m} \tag{A.32}$$

The percentage-increment ratios I_{xT} and I_{xR} derived from equation (A.32) are given by:

$$I_{xT} = 2 \tag{A.33}$$

$$I_{xR} = - \frac{1}{1 - C_R} \tag{A.34}$$

The corresponding expressions for I_{xm}, I_{xB}, I_{WT}, I_{WR}, I_{Wm} and I_{WB} can be obtained from equations (A.17), (A.18), (A.22) and (A.23). Similarly the corresponding expressions for triangular loads can be derived. These expressions are summarized in paragraph A-08b.

The percentage-increment ratios for both x_m and W_m of a completely plastic system subjected to rectangular and triangular loads are given in figures A.3 and A.4 respectively. From these two figures, the following conclusions can be drawn:

(1) For both types of loads, I_{xT} and I_{WT} are equal to 2, and I_{xm} and I_{Wm} are equal to unity. Hence, for a completely plastic system subjected to simplified loads the maximum displacement x_m and the maximum work done W_m are: (1) proportional to the square of load duration T and (2) inversely proportional to the mass m.

(2) The numerical values of I_{xB} or I_{WB} are always greater than 2 and the numerical value of I_{xR} is always greater than unity. When $C_R < 0.3$, I_{xB} and I_{xR} are approximately equal to 2 and -1 respectively.

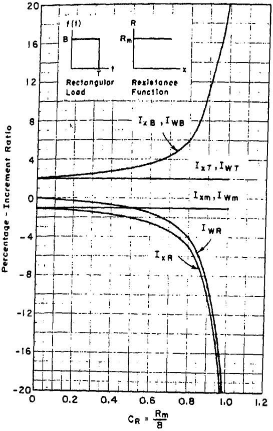

Figure A.3. Percentage increment ratios for a completely plastic system subjected to rectangular loads

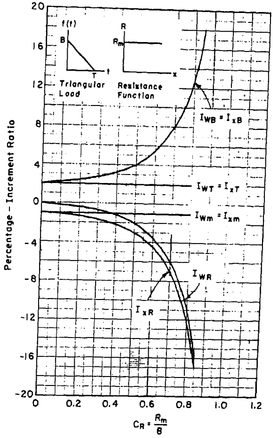

Figure A.4. Percentage increment ratios for a completely plastic system subjected to triangular loads

Hence, for such cases the maximum displacement is proportional to the square of the peak load, B , and inversely proportional to the plastic resistance, R_m .

(3) When $C_R > 0.5$, for rectangular loads, $I_{xB} > 3$ and $I_{xR} < -2$; and for triangular loads, $I_{xB} > 4$ and $I_{xR} < -3$. Thus the maximum displacement is extremely sensitive to variations in B and R_m . For such cases it would be meaningless, in a trial and error design procedure, to seek an agreement better than ±30% between the maximum displacement obtained by a design computation and that of the design specification.

A-06 PERCENTAGE-INCREMENT RATIOS FOR ELASTO-PLASTIC SYSTEMS. The expressions for the percentage-increment ratio of an elasto-plastic system subjected to simplified loads can be derived by the methods described in paragraphs 5-07d, A-02 and A-03. An example is used to illustrate the procedure in the derivation. The independent variables of the dynamic system chosen as example satisfy the following conditions: (1) the load is a triangular load; (2) the displacement reaches the limiting elastic displacement x_e at time t_e where $t_e < T$; and (3) the maximum displacement is reached at t_m where $t_m < T$.

When $t < t_e$, the equation of motion is given by equation (5-33a). When $t_e < t < t_m$, the equation of motion is given by equation (5-33b). The expression for x_m derived from these two differential equations is given by equation (A.35).

$$x_m = \frac{R_m}{k} + v_e\left(t_m - t_e\right) + \frac{B}{m}\left(1 - \frac{t_e}{T}\right)\left(t_m - t_e\right)^2\left[\frac{1}{2} - \frac{t_m - t_e}{6\left(T - t_e\right)}\right] - \frac{R_m}{2m}\left(t_m - t_e\right)^2 \qquad (A.35)$$

where

v_e = the velocity of the mass at time t_e.

This equation shows that x_m is a function of B, T, m, k, R_m, t_m, t_e and v_e. The last three variables: t_m, t_e, and v_e, in turn, depend on B, T, m, k, and R_m. Considering the effect of load duration T on x_m, the general expression of I_{xT}, instead of being given by equation (A.3), is given by equation (A.36).

$$I_{xT} = \frac{T}{x_m}\left[\frac{\partial x_m}{\partial T} + \frac{\partial x_m}{\partial t_e}\frac{\partial t_e}{\partial T} + \frac{\partial x_m}{\partial t_m}\frac{\partial t_m}{\partial T} + \frac{\partial x_m}{\partial v_e}\frac{\partial v_e}{\partial T}\right] \qquad (A.36)$$

After the partial derivatives in equation (B.36) are evaluated, the expression of I_{xT} is given by:

$$I_{xT} = -1 + \frac{1}{C_x}\left\{2\pi^2 C_T^2\left(\frac{1}{C_R} - 1\right)\left(\frac{t_m}{T} - \frac{t_e}{T}\right)^2 + \frac{2\pi C_T}{C_R}\left(\frac{t_m}{T} - \frac{t_e}{T}\right)\sin\left(2\pi\frac{t_e}{T_n}\right) + \frac{1}{C_R}\left(1 - \cos 2\pi\frac{t_e}{T_n}\right)\right\} \qquad (A.37)$$

By a similar procedure, the expressions for all the percentage-increment ratios of an elasto-plastic system subjected to triangular or rectangular loads can be derived. These expressions are summarized in paragraph A-08c.

The percentage-increment ratios for elasto-plastic systems subjected to rectangular and triangular loads are given in figures A.5 through A.8. The first two figures are for rectangular loads and the last two, for triangular loads. In figures A.5 and A.7, the percentage-increment ratios with respect to T, m and k are plotted against C_T with C_R as the running parameter. In figures A.6 and A.8, the independent variables are B and R_m. From these four figures, the following conclusions can be drawn:

(1) The effect of load duration T. The curves in figures A.5 and A.7 are divided into two groups by the curve corresponding to $C_R = 1$. When $C_R < 1$, the value of I_{xT}, with a few exceptions, lies between 1 and 2. Hence for a given C_R, the maximum displacement is proportional to T at small values of C_T and is proportional to the square of T at large values of C_T. The smaller the values of C_R, the smaller will be the value of C_T when I_{xT} approaches 2. On the other hand when $C_R > 1.0$, the value of I_{xT} lies approximately between zero and unity. Hence, at small values of C_T, the maximum displacement is approximately proportional to the load duration T, while at large values of C_T, the maximum displacement is practically independent of the load duration T.

(2) The effect of the mass m and the spring constant k on the maximum displacement can be approximately summarized into the following two statements. (1) When $C_R < 1$, the values of I_{xm} lie approximately between -0.5 and -1.0, and the values of I_{xk} lie between -0.5 and 0. Hence, at small values of C_T, the maximum displacement is inversely proportional to the square root of both the mass, m, and the spring constant, k. At large values of C_T, x_m is inversely proportional to m but is independent of k. (2) When C_R 1.0. At small values of C_T, x_m is inversely proportional to square root of both m and k. But at large values of C_T, x_m is inversely proportional to k but is independent of m.

(3) The effect of the peak load B and the plastic resistance R_m

As shown in figures A.6 and A.8, the numerical values of I_{xB} and I_{xR} scatter over a much wider range than the corresponding values of I_{xT},

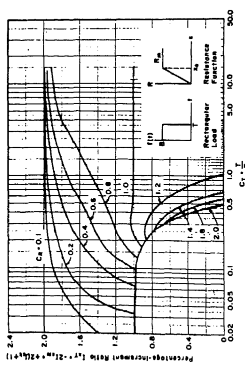

Figure A.5. Percentage-increment ratios I_{xT}, I_{xk} for an elasto-plastic system subjected to rectangular loads

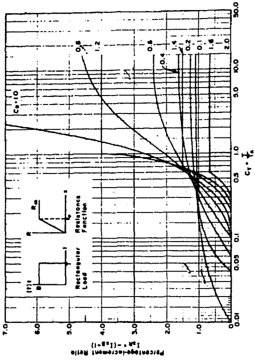

Figure A.6. Percentage-increment ratios I_{xR} and I_{xB} for an elasto-plastic system subjected to rectangular loads

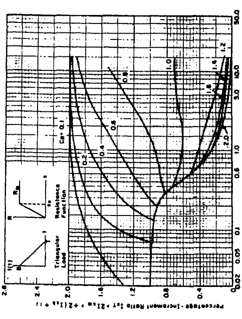

Figure A.7. Percentage-increment ratios I_{xT}, I_{xm} and I_{xk} for an elasto-plastic system subjected to triangular loads

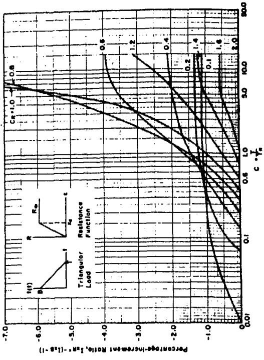

Figure A.8. Percentage-increment ratios I_{xR} and I_{xB} for an elasto-plastic system subjected to triangular loads

I_{xm} or I_{xk} . For the convenience of discussion, the curves in these two figures can be very roughly divided into three regions. (1) The first region covers all cases when $C_T < 0.6$. Practically for all values of C_R, the values of I_{xB} lie between 1 and 2, and the values of I_{xR} lie between 0 and -1. Thus for low values of C_T, x_m is directly proportional to B but independent of R_m; while for high values of C_T, x_m is proportional to the square of B and inversely proportional to R_m. The parameters of one story buildings are generally located in this region. Since the numerical values of both I_{xB} and I_{xR} are not very large, a close agreement between the maximum displacement obtained by a design computation and that of the design specification can generally be achieved. (2) The second region corresponds to cases when $C_T > 1.0$ and $C_R > 1.4$ or $C_R < 0.4$. In this region both I_{xB} and I_{xR} are practically constant. The numerical values of neither one is very large. The cases when $C_R < 0.4$ are not of practical importance because the corresponding values of C_x are always greater than 20. On the other hand, the cases when $C_R > 1.4$ generally correspond to the region in which the design of structural members such as beams, walls or columns are located. (3) The third region corresponds to cases when $C_T > 1.0$ and $0.4 < C_R < 1.4$. In this region I_{xB} is generally greater than 2.5 and I_{xR} is less than -1.5. Therefore, the maximum displacement x_m is very sensitive to any variation in B or R_m especially when C_R lies between 0.8 and 1.2. In design problems, both B and R_m are subject to uncertainties in their determinations. Therefore the value of x_m obtained by a design computation would not be very reliable. Hence it would be a good design practice, in general, to select the parameter of a design such that the combination of C_T and C_R lies outside of the third region.

A-07 THE EFFECT OF LOAD SHAPE ON THE RESPONSE OF DYNAMIC SYSTEMS. The effect of load shape is partially investigated in EM 1110-345-415 where the responses of a dynamic system subjected to different types of load, as shown in figure 5.8, are analyzed and presented separately. For each type of load with a given geometry, i.e., a triangle or a rectangle, its "shape" or "proportion" is completely specified by two variables. These are the peak load B and the load duration T . When either B or T is varying, the response is also varying continuously. Therefore, in paragraph A-02, partial derivatives of x_m with respect to B or T can be evaluated and the effect of B and T can be expressed quantitatively in terms of percentage-increment ratios. Here again the percentage-increment ratios for rectangular and triangular loads are considered and presented separately.

If the concept of percentage-increment ratio is to be extended to the analysis of the effect of load shape on the response, it would not be sufficient to consider just discrete types of load, for example rectangular and triangular loads. Instead, another variable has to be added to the load pulse so as to form a family of load curves, and the geometrical shape of the load curve will then be varying in a continuous fashion. For example, in figure A.9, when T_1 varies from 0 to T, the geometrical load shape varies continuously from a triangular load to a rectangular load. A percentage-increment ratio I_{xT_1} could then be defined which expresses the relative magnitude of the percentage changes in x_m and T_1. And the effect of changing the geometrical shape of the load on the response could then be analyzed quantitatively.

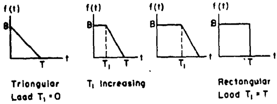

Figure A.9. A set of load curves with T_1 as a parameter

It is clear that a rigorous investigation of the effect of the load shape cannot be carried out without introducing another variable. For an elasto-plastic system, there would be 4 non-dimensional variables involved and the result could not be conveniently presented in graphical form. Although for linearly elastic or completely plastic systems there would be no difficulty in presenting I_{xT_1} for a given family of load shapes. Its practical application would be limited to the case where the actual load shape variation follows the same pattern as the variation in the given family. Because of these difficulties, as far as the effect of load shape is concerned, a qualitative discussion, supplemented by non-dimensional graphs whenever convenient, is presented in this paragraph. Specifically, two cases are considered. In the first case, the response of an elasto-plastic system to rectangular and triangular loads is compared with each other. This will indicate in what region, if any, the load shape is not important. In the second case, the percentage-increment ratio I_{xJ}, given by equation (A.28), for rectangular and triangular loads is presented in graphical form. These curves will show how the response is affected by changing the "shape" or "proportion" of the load curve without changing its geometry.

a. Comparison of the Responses of an Elasto-Plastic System Subjected to Rectangular and Triangular Loads. In figure A.10, the ratios of the maximum displacement for an elasto-plastic system subjected to a rectangular load to the corresponding value for a triangular load are plotted. There are three regions in this figure where the shape of the load pulse has no

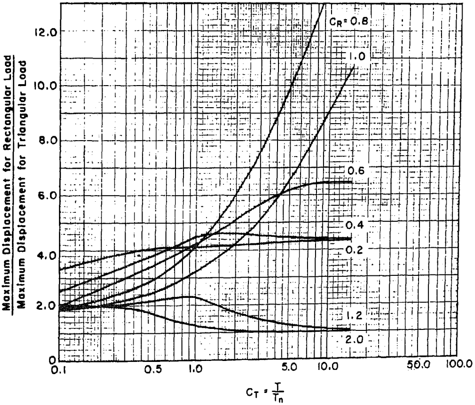

Figure A.10. Ratio of maximum displacements for rectangular to the corresponding values in triangular load, elasto-plastic system

important effect on the response. (1) In the first region, where $C_T > 0.5$ and $C_R < 0.4$, the ratio of x_m's is approximately equal to 4. The corresponding values of C_x and $\frac{t_m}{T}$ are always greater than unity. In this region, the maximum strain energy is approximately proportional to x_m but the maximum work done is approximately proportional to the square of the area under the load-time curve. Therefore, the maximum displacement is approximately proportional to the square of the area under the load curve while the detail geometry of the load curve is not important. (2) In the second region where $C_T < 0.5$ and $C_R > 1.2$, the ratio of x_m's is approximately equal to 2. In this region, $x_m < 2x_e$, and $t_m > T$. Although W_m is still approximately proportional to the square of the area of the load pulse, the maximum strain energy is also proportional to the square of x_m . There the maximum displacement is proportional to the area under the load-time curve and is not much affected by the detail shape of the load. (3) In the third region where $C_T > 2$ and $C_R > 1.2$, the ratio of x_m's is approximately equal to unity. This corresponds to the cases where t_m is small in comparison with T . The maximum displacement is proportional to the peak load B while the load shape is not important.

Outside of the above three regions, the ratios of x_m's for rectangular and triangular loads vary considerably with both C_T and C_R . No simple conclusion can be drawn. Generally the response is expected to be affected by the detail geometry or the load curve.

b. The Percentage-Increment Ratio I_{xJ} . The percentage-increment ratio I_{xJ} given by equation (A.28) is plotted in figures A.11 and A.12 for rectangular and triangular loads respectively. As discussed in paragraph A-03c, the value of I_{xJ} indicates the effect of varying the "shape" or "proportion" of a load curve without changing its basic geometry. In figures A.11 and A.12, the numerical values of I_{xJ} are small when $C_T < 0.3$. In this region the response will not be much affected when one triangular load is replaced by another triangular load, or one rectangle by another rectangle, or any load curve which resembles a triangle by the corresponding triangular load.

A-08 EXPRESSIONS FOR MAXIMUM DISPLACEMENTS AND PERCENTAGE-INCREMENT RATIOS. The expressions of t_m, x_m, I_{xR} and I_{xT} for linearly elastic,

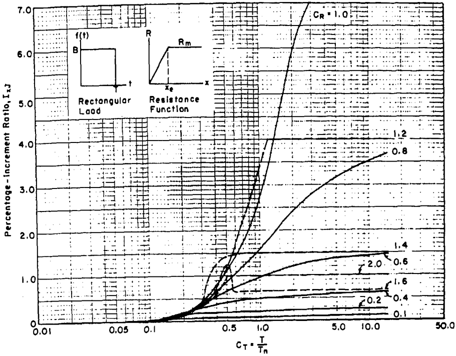

Figure A.11. The percentage-increment ratio I_{xS} for an elasto-plastic system subjected to rectangular loads

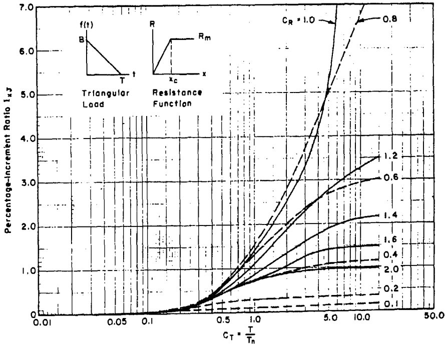

Figure A.12. The percentage-increment ratio I_{xS} for an elasto-plastic system subjected to triangular loads

completely plastic and elasto-plastic systems subjected to rectangular and triangular loads are given in this paragraph. From the expressions of these four quantities, the expressions of many other quantities can be derived. For example, in terms of x_m, the expression for W_m is:
when $x_m < x_e$

$$W_m = \frac{1}{2} k x_m^2 \tag{A.38}$$

and when $x_m > x_e$

$$W_m = R_m \left(x_m - \frac{x_e}{2} \right) \tag{A.39}$$

In terms of I_{xT} and I_{xR}, the expressions of many other percentage-increment ratios can be obtained. These relationships are given by equations (A.40) and (A.41).

$$I_1 = I_{xT} = -I_{xm} = 2\left(I_{xk} - 1\right) = \frac{2C_x - 1}{2C_x} I_{WT} = \frac{1 - 2C_x}{C_x} I_{Wm} = \frac{2C_x - 1}{C_x} \left(I_{Wk} + 1\right) \tag{A.40}$$

$$I_2 = I_{xR} - 1 = -I_{xB} = \frac{1 - 2C_x}{2C_x} I_{WB} = \frac{2C_x - 1}{2C_x} \left(I_{WR} - 2\right) \tag{A.41}$$

a. Expressions for Linearly Elastic Systems.

Rectangular Loads

(i) For $t_m < T$

$$\frac{t_m}{T} = \frac{1}{2} \tag{A.42a}$$

$$\text{D.L.F.} = \frac{k x_m}{B} = 2 \tag{A.42b}$$

where

D.L.F. = dynamic load factor

$$I_{xT} = 0 \tag{A.42c}$$

$$I_{xR} = 0 \qquad \text{(A.42d)}$$

(ii) For $t_m > T$

$$\frac{t_m}{T_n} = \frac{1}{2\pi} \tan^{-1} \left[\frac{\sin 2\pi C_T}{\cos 2\pi C_T - 1} \right] \qquad \text{(A.43a)}$$

$$\text{D.L.F.} = 2 \sin \left(\pi C_T \right) \qquad \text{(A.43b)}$$

$$I_{xT} = \pi C_T \cot \left(\pi C_T \right) \qquad \text{(A.43c)}$$

$$I_{xR} = 0 \qquad \text{(A.43d)}$$

Triangular Loads

(i) For $t_m < T$

$$\frac{t_m}{T_n} = \frac{1}{\pi} \tan^{-1} \left(2\pi C_T \right) \qquad \text{(A.44a)}$$

$$\text{D.L.F.} = 2 \left[1 - \frac{\tan^{-1} 2\pi C_T}{2\pi C_T} \right] \qquad \text{(A.44b)}$$

$$I_{xT} = -1 + \left[\frac{4\pi^2 C_T^{\,2}}{1 + 4\pi^2 C_T^{\,2}} \right] \left[\frac{2\pi C_T}{2\pi C_T - \tan^{-1} \left(2\pi C_T \right)} \right] \qquad \text{(A.44c)}$$

$$I_{xR} = 0. \qquad \text{(A.44d)}$$

(ii) For $t_m > T$

$$\frac{t_m}{T_n} = \frac{1}{2\pi} \tan^{-1} \left[\frac{1 - \cos 2\pi C_T}{\sin \left(2\pi C_T \right) - 2\pi C_T} \right] \qquad \text{(A.45a)}$$

$$\text{D.L.F.} = \frac{1}{2\pi C_T} \left[2 + 4\pi^2 C_T^{\,2} - 2 \cos \left(2\pi C_T \right) - 4\pi C_T \sin \left(2\pi C_T \right) \right]^{\frac{1}{2}} \qquad \text{(A.46b)}$$

$$I_{xT} = -1 + \frac{1 - \cos \left(2\pi C_T \right)}{\text{D.L.F.}} \qquad \text{(A.46c)}$$

$$I_{xR} = 0 \qquad \text{(A.46d)}$$

b. Expressions for Completely Plastic Systems.

Rectangular Load

$$\frac{t_m}{T} = \frac{1}{C_R} \qquad (A.47a)$$

$$\frac{x_m}{\frac{1}{2}\frac{B}{m}T^2} = \frac{1 - C_R}{C_R} \qquad (A.47b)$$

$$I_{xT} = 2 \qquad (A.47c)$$

$$I_{xR} = -\frac{1}{1 - C_R} \qquad (A.47d)$$

Triangular Load

(i) For $C_R > 0.5$ and $t_m < T$

$$\frac{t_m}{T} = 2\left(1 - C_R\right) \qquad (A.48a)$$

$$\frac{x_m}{\frac{1}{2}\frac{B}{m}T^2} = \frac{4}{3}\left(1 - C_R\right)^3 \qquad (A.48b)$$

$$I_{xT} = 2 \qquad (A.48c)$$

$$I_{xR} = \frac{3C_R}{1 - C_R} \qquad (A.48d)$$

(ii) For $C_R < 0.5$ and $t_m > T$

$$\frac{t_m}{T} = \frac{1}{2\,C_R} \qquad (A.49a)$$

$$\frac{x_m}{\frac{1}{2}\frac{B}{m}T^2} = \left(\frac{1}{4\,C_R} - \frac{1}{3}\right) \qquad (A.49b)$$

$$I_{xT} = 2 \qquad (A.49c)$$

$$I_{xR} = \frac{3}{3 - 4\,C_R} \qquad (A.49d)$$

c. Expressions for Elasto-Plastic Systems.

Rectangular Load

(i) For $x_m < x_e$ and $t_m < T$. The expressions for this case are the same as those given by equation (A.42).

(ii) For $x_m < x_e$ and $t_m > T$. The expressions for this case are the same as those given by equation (A.43).

(iii) For $x_m > x_e$, $t_m > T$ and $t_e > T$.

$$\cos\left(2\pi \frac{t_e}{T_n}\right) = - C_R + \cos 2\pi\left(\frac{t_e}{T_n} - \frac{T}{T_n}\right) \tag{A.50a}$$

$$\frac{t_m}{T_n} = \frac{t_e}{T_n} + \frac{1}{2\pi C_R}\left[\sin 2\pi \frac{t_e}{T_n} - \sin 2\pi\left(\frac{t_e}{T_n} - \frac{T}{T_n}\right)\right] \tag{A.50b}$$

$$C_x = \frac{x_m}{x_e} = \frac{1}{2} + \frac{1}{C_R^2}\left[1 - \cos 2\pi C_T\right] \tag{A.50c}$$

$$I_{xT} = \frac{C_R^2}{C_x}\left[2\pi C_T\right] \sin\left(2\pi C_T\right) \tag{A.50d}$$

$$I_{xR} = \frac{1}{C_x} - 1 \tag{A.50e}$$

(iv) For $x_m > x_e$, $t_e < T$ and $t_m < T$.

$$\frac{t_e}{T_n} = \frac{1}{2\pi} \sin^{-1} \sqrt{\frac{C_R}{2}} \tag{A.51a}$$

$$\frac{t_m}{T_n} = \frac{1}{2\pi} \sin^{-1} \sqrt{\frac{C_R}{2}} - \frac{1}{2\pi} \frac{\sqrt{C_R\left(2 - C_R\right)}}{1 - C_R} \tag{A.51b}$$

$$\frac{x_m}{x_e} = \frac{C_R}{2\left(C_R - 1\right)} \tag{A.51c}$$

$$I_{xT} = 0 \tag{A.51d}$$

$$I_{xR} = \frac{C_R - 2}{C_R - 1} \tag{A.51e}$$

(v) For $x_m > x_e$, $t_e < T$ and $t_m > T$.

$$\frac{t_e}{T_n} = \frac{1}{2\pi} \sin^{-1} \sqrt{\frac{C_R}{2}} \tag{A.52a}$$

$$\frac{t_m}{T_n} = \frac{1}{C_R}\left[C_T + \sin 2\pi \frac{t_e}{T_n} - 2\pi \left(1 - C_R\right) \frac{t_e}{T_n}\right] \tag{A.52b}$$

$$\frac{x_m}{x_e} = \frac{1}{2} + \frac{1}{C_R} + 2\pi \frac{\sqrt{C_R \left(2 - C_R\right)}}{C_R^2} \left(C_T - \frac{t_e}{T_n}\right) + \frac{2\pi^2 \left(1 - C_R\right)}{C_R^2} \left[C_T - \frac{t_e}{T_n}\right]^2 \tag{A.52c}$$

$$I_{xT} = 2\pi \frac{C_T}{C_x C_R^2} \left\{\sqrt{C_R \left(2 - C_R\right)} + 2\pi \left(1 - C_R\right) \left(C_T - \frac{t_e}{T_n}\right)\right\} \tag{A.52d}$$

$$I_{xR} = -1 + \frac{1}{C_x} - \frac{2\pi^2 \left(C_T - \frac{t_e}{T_n}\right)^2}{C_x C_R^2} \tag{A.52e}$$

Triangular Load

(i) For $x_m < x_e$, and $t_m < T$. The expressions are the same as those given by equation (A.44).

(ii) For $x_m < x_e$, and $t_m > T$. The expressions are the same as those given by equation (A.45).

(iii) For $x_m > x_e$, $t_e > T$, and $t_m > T$.

$$\sin 2\pi \frac{t_e}{T_n} = 2\pi C_R C_T + 2\pi C_T \cos \left(2\pi \frac{t_e}{T_n}\right) + \sin 2\pi \left(\frac{t_e}{T_n} - C_T\right) \tag{A.53a}$$

$$\frac{t_m}{T_n} = \frac{t_e}{T_n} + \frac{1}{4\pi^2 C_R C_T} \left\{\cos 2\pi \frac{t_e}{T_n} + 2\pi C_T \sin \left(2\pi \frac{t_e}{T_n}\right) - \cos 2\pi \left(\frac{t_e}{T_n} - C_T\right)\right\} \tag{A.53b}$$

$$\frac{x_m}{x_e} = 1 + 2\pi^2 \left(\frac{t_m}{T_n} - \frac{t_e}{T_n}\right)^2 \tag{A.53c}$$

$$I_{xT} = -1 - \frac{2\pi^2}{C_x} \left(\frac{t_m}{T_n} - \frac{t_e}{T_n}\right)^2 + \frac{C_R^2}{C_x} \left[1 - \cos 2\pi C_T\right] \tag{A.53d}$$

$$I_{xR} = -\frac{2\pi^2}{C_x} \left(\frac{t_m}{T_n} - \frac{t_e}{T_n}\right)^2 \tag{A.53e}$$

(iv) For $x_m > x_e$, $t_e < T$, and $t_m < T$.

$$\frac{t_e}{T} = 1 - C_R - \frac{1}{2\pi C_T} \sin\left(2\pi \frac{t_e}{T_n}\right) - \cos\left(2\pi \frac{t_e}{T_n}\right) \qquad \text{(A.54a)}$$

$$\frac{t_m}{T} = \frac{t_e}{T} + \frac{1}{C_R}\left[1 - \frac{t_e}{T}\right]\left[\frac{t_m}{T} - \frac{t_e}{T} - \frac{\left(\frac{t_m}{T} - \frac{t_e}{T}\right)^2}{2\left(1 - \frac{t_e}{T}\right)}\right] + \frac{1}{4\pi^2 C_T^2 C_R}\left[- 1 + \cos 2\pi \frac{t_e}{T_n} + 2\pi C_T \sin\left(2\pi \frac{t_e}{T_n}\right)\right] \qquad \text{(A.55b)}$$

$$\frac{x_m}{x_e} = 1 + \frac{4\pi^2}{3}\frac{C_T^2}{C_R}\left(\frac{t_m}{T} - \frac{t_e}{T}\right)^3 + 2\pi^2\left(\frac{t_m}{T_n} - \frac{t_e}{T_n}\right)^2 - \frac{2\pi^2}{C_R}\left(1 - \frac{t_e}{T}\right)\left(\frac{t_m}{T_n} - \frac{t_e}{T_n}\right)^2 \qquad \text{(A.55c)}$$

$$I_{xT} = -1 + \frac{1}{C_x}\left\{\left[\frac{1}{C_R} - 1\right] 2\pi^2\left(\frac{t_m}{T_n} - \frac{t_e}{T_n}\right)^2 + \frac{2\pi}{C_R}\left(\frac{t_m}{T_n} - \frac{t_e}{T_n}\right)\sin\left(2\pi \frac{t_e}{T_n}\right) + 1 - \cos\left(2\pi \frac{t_e}{T_n}\right)\right\} \qquad \text{(A.55d)}$$

$$I_{xR} = - \frac{2\pi^2}{C_x}\left(\frac{t_m}{T_n} - \frac{t_e}{T_n}\right)^2 \qquad \text{(A.55e)}$$

(v) For $x_m > x_e$, $t_e < T$ and $t_m > T$.

$$\frac{t_e}{T} = 1 - C_R - \frac{1}{2\pi C_T} \sin\left(2\pi \frac{t_e}{T_n}\right) - \cos\left(2\pi \frac{t_e}{T_n}\right) \qquad \text{(A.56a)}$$

$$\frac{t_m}{T_n} = \frac{t_e}{T_n} + \frac{1}{4\pi^2 C_R C_T}\left\{2\pi^2\left(C_T - \frac{t_e}{T_n}\right)^2 + 2\pi C_T \sin\left(2\pi \frac{t_e}{T_n}\right) - 1 + \cos 2\pi \frac{t_e}{T_n}\right\} \qquad \text{(A.56b)}$$

$$\frac{x_m}{x_e} = 1 + 2\pi^2 \left[\frac{t_m}{T_n} - \frac{t_e}{T_n}\right]^2 - \frac{2\pi^2\, C_T^{\,2}}{3\, C_R} \left[1 - \frac{t_e}{T}\right]^3 \qquad \text{(A.56c)}$$

$$I_{xT} = -1 + \frac{1}{C_x} \left\{ -2\pi^2 \left[\frac{t_m}{T_n} - \frac{t_e}{T_n}\right]^2 + \frac{1}{C_R^{\,2}} \left[2\pi^2 C_T^{\,2} \left(1 - \frac{t_e}{T}\right)^2 \cos\left(2\pi \frac{t_e}{T_n}\right) + 1 - \cos\left(2\pi \frac{t_e}{T_n}\right) + 2\pi C_T \left(1 - \frac{t_e}{T}\right) \sin\left(2\pi \frac{t_e}{T_n}\right) \right] \right\} \qquad \text{(A.56d)}$$

$$I_{xR} = -\frac{2\pi^2}{C_x} \left[\frac{t_m}{T_n} - \frac{t_e}{T_n}\right]^2 \qquad \text{(A.56e)}$$

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